ARITHMETIC

FOR HIGH SCHOOLS, ACADEMIES AND NORMAL SCHOOLS

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BY

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"Numbre is the onelie thing that separateth man from the beastes. Hee therefore that shall contempne Numbre, hee declareth himselfe as brutishe as a beaste, and unworthy to be counted in the fellowshippe of men. But I truste there is no man so foule ouerseene, though manie right smallye do it regarde." — Recorde.

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PREFACE

In the preparation of this work it is assumed that the learner has gained facility in performing the four fundamental operations of Arithmetic, — Addition, Subtraction, Multiplication, and Division. This facility should be strongly insisted upon, since the success hoped for throughout Arithmetic is dependent upon it.

The subjects included in the present work are those which have usually found place in the leading schools, and yet there is more offered than can be completed in an average term. The teacher is expected to use his judgment in regard to omissions.

The contents of this book have been tested by the author with classes in the high school and in the normal school. The pupils who have pursued the work conscientiously have met the requirements expected of them.

There is no attempt either to introduce novel methods or any new arrangement of topics. There are no puzzles offered. They usually have no well-established principles behind them.

Throughout the work are scattered historical facts which it is hoped will prove interesting to intelligent students. It is thought this is a feature of the book.

Much effort has been put forth to insure accuracy in this respect.

Attention is called to other features of the book, such as the treatment of Ratio, and of the Applications of Percentage, especially that of Stocks and Bonds.

While care has been taken to insure against inaccuracies, both as regards numerical calculation and history, it can hardly be hoped that the work is free from error. Therefore thoughtful suggestions and corrections will be thankfully received.

I desire to offer my sincerest thanks to those who have assisted me with advice and criticism; especially to Professors Frank R. Higgins and W. P. Morgan of the Indiana State Normal School, and Professor C. T. Lane of the Fort Wayne High School.

O. L. K.

TERRE HAUTE,
February, 1903.

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ARITHMETIC

The word "Arithmetic" is derived from the Greek word *arithmos*, number, and designates the study of number.

ORIGIN OF NUMBER

The primitive conceptions of number were very crude. The origin of the conceptions is uncertain. Careful investigation has failed to yield knowledge sufficiently accurate to justify a discussion of this subject in a work like the present.

The origin of the conceptions of number is probably beyond the proper limits of inquiry.

Our ideas of number are the results of our past experiences and "those of our ancestors."

The reader is referred to "The Number Concept — Its Origin and Development," by Levi L. Conant, The Macmillan Co.; "The Number System of Algebra," by H. B. Fine, D. C. Heath & Co.; and "The Psychology of Number," by McLellan and Dewey, D. Appleton & Co.

DEFINITIONS OF NUMBER

Euclid: "A number is a collection or an assemblage of things of the same kind."

This definition seems faulty, since things may be collected or assembled without the number idea, and the

number idea may exist without things being collected or assembled. But this definition has been used for centuries.

Newton: "Number is the abstract ratio of one quantity to another quantity of the same kind."

This is a very good definition; yet the word "abstract" need not appear, since all ratio is abstract.

Euler: "Number is the ratio of one quantity to another of the same kind taken as a unit."

Glashan: "Number is that which is applied to a unit to express the comparative magnitude of a quantity of the same kind as the unit."

The definitions of Euclid, Newton, and Euler have stood the test for centuries, and the authors were men of undoubted mathematical insight.

Definitions of masters in mathematics are here placed before the student. Careful study and thorough mastery of them will be a better exercise in clear thinking than an attempt by the student to make a definition for himself, unless he has acquired a power of research that is quite unusual among students of the age and training of those who will use this book.

De Morgan says "it is not possible to define what we mean by number and quantity." In so doing he thinks "we are guilty of the absurdity of attempting to make the simplest ideas yet more simple."

Glashan is a modern writer, and has given us a good definition without using the word "ratio."

COUNTING

The whole of Arithmetic is the result of a very simple act — that of *counting*.

Counting gives rise to the infinite series of positive integral numbers.

When man begins to count he cannot think of number abstractly; he cannot think of number away from the objects themselves. When he uses his fingers to assist him in counting other objects, then the real idea of number is beginning to unfold to him.

Sometimes he uses pebbles or strokes in the sand to assist him, and then the real idea is further asserting itself.

When man reckons with his fingers, with pebbles, or with strokes in the sand, each finger, or pebble, or stroke may represent one or more things. In so far as they are used to count other objects they are symbols of number; as much so as the Roman "I" or the Arabic "5."

When he can represent the number of objects in a group by an equal number of other objects differing in many respects, he has gained the power to think of the number of objects in a group *away* from the objects themselves.

"Counting leads to an expression for the number of things in any group in terms of a representative group." This representative group may be the fingers, pebbles, marks in the sand, a numeral word, or a numeral symbol in common use.

There is a difference between counting by use of pebbles or marks in the sand and by the use of the common numeral symbols, in that there is no order in the one such as is found in the other. Yet the South Sea Islanders met this distinction in counting by use of cocoanut stalks. They put "aside a little one when they counted *ten*, and a large one when they counted *one hundred*."

Many attempts have been made to agree upon a fixed set of symbols which should serve as a common medium for conveying number ideas and for calculations.

It is the distinctive province of Arithmetic to calculate or compute by means of numerical characters, and the

worth of a system of notation is to be estimated by the facilities it affords for carrying on mathematical calculations.

EXPRESSION OF NUMBER

The Gesture Method. No doubt the earliest method of representing numbers was the finger method, the fingers being used to designate the objects whose number was desired. This method is frequently used among the masses to this day for communicating small numbers, and is used by brokers on the stock exchange. Since the fingers are always accessible; since they are "so similar in form and in function"; and since the system of expressing number most in use (the Arabic) is decimal, the method of expressing number by the fingers probably antedated all other methods.

However, it is a fact that some tribes do not know numbers beyond the number 2; this tends to support the theory that possibly some other method existed prior to the finger method. With the fingers alone, or with any concrete objects, it would be difficult to represent $\frac{16}{17}$ of an apple directly.

The Chinese Method. It is relatively easy to count up to ten; but primitive people have found difficulty in counting higher numbers. In South Africa it takes two people to count higher numbers, one to count up to ten and another to count and keep record of these tens. But the Chinese overcame this great difficulty by the use of the *swan-pan.* This instrument consists of a rectangular board surrounded by a frame in which are fastened ten rods made of bamboo, on which are strung small ivory balls. All numbers up to ten are recorded on one rod; the tens are recorded on a second rod, and so on. It is at

once seen that numbers almost without limit can be represented by the *swan-pan*.

The Word Method. This method followed the gesture method as soon as the language of a people was sufficiently developed to have words for numbers; as, *one*, *two*, *five*, *twelve*, etc.

The Roman Method. All are somewhat familiar with the Roman method and how it employs the letters of the alphabet to represent numbers. Why the Romans used some of the letters to the exclusion of the others is not so well known. It is thought that the Romans did not invent this system, but that the Etruscans did. However, the Romans practised the system.

This system employs eight characters in the expression of numbers: the seven letters I, V, X, L, C, D, and M, and the bar (—). I stands for one, V for five, X for ten, L for fifty, C for one hundred, D for five hundred, M for one thousand; and the bar (—), when placed over any letter or group of letters, multiplies the value of the number thus expressed by one thousand; thus, $\overline{\text{V}}$ stands for five thousand, $\overline{\text{L}}$ for fifty thousand, $\overline{\text{XC}}$ for ninety thousand.

The positional idea is present in the Roman System to a very limited extent; as, VI, IV, IX, and LX. For the most part the value attached to a symbol depends upon its shape and not upon its place or position.

The following principles govern the Roman System:

I. Repeating a letter repeats its value; II stands for two; XX for twenty; CC for two hundred.

II. When a letter is placed *after* one of greater value, the number represented is the sum of the numbers expressed by the letters separately; VI stands for five and one, or six; LX for sixty; CI for one hundred one; DV for five hundred five.

III. When a letter is placed *before* one of greater value, the number expressed is the difference of the numbers represented by the letters singly; IV stands for four; XL for forty; and XC for ninety. VX is not allowable, for the value represented has a symbol of its own. The same might be said of DM and LC. IIX stands for eight.

IV. When a letter stands between two of greater values, its value is combined with that of the one following it, or its value is taken from the combined values of the two greater; XIV stands for fourteen; CXL for one hundred forty; and LIV for fifty-four.

V. A letter is placed before one of its own order, or before one of the next higher order only. Hence, IM was not used for nine hundred ninety-nine, for I and M do not stand in consecutive, nor the same orders. IL was not used for forty-nine, nor XD for four hundred ninety.

V, L, and D do not bear repetition. VV, LL, and DD represent numbers which have symbols of their own.

At the present day the Roman System of expressing numbers is not used for computation, but is used for marking the number of a chapter, a page of a book, the time on the face of a clock, or physicians' prescriptions.

EXERCISES

Express the following by the Roman Notation:

1. 18.	6. 134.	11. 1846.	16. 15,625.
2. 35.	7. 249.	12. 3497.	17. 40,317.
3. 89.	8. 574.	13. 6493.	18. 93,807.
4. 67.	9. 783.	14. 8002.	19. 134,615.
5. 96.	10. 490.	15. 9031.	20. 568,074.

Express the following in the Arabic Notation:

21.	XIX.	25.	DLXI.	29.	$\overline{\mathrm{V}}\mathrm{IV}$.
22.	LIX.	26.	DCLXVI.	30.	$\overline{\mathrm{LV}}\mathrm{LXXV}$.
23.	CXIV.	27.	DCXLIV.	31.	$\overline{\overline{\mathrm{C}}\ \mathrm{LIX}}\ \mathrm{DLXXXVI}$.
24.	CLXVI.	28.	DLXVI.	32.	$\overline{\mathrm{M}}\ \overline{\mathrm{DIV}}\ \mathrm{CCCXIV}$.

The Arabic Method. This is the method that is in common use. It is superior to any other method in existence, since it is the clearest and simplest. To appreciate its superiority one has but to attempt a problem in division by some other method.

It is called the Arabic Method, or System, because the Arabs introduced it into Europe, although the system is probably of Indian origin.

The Arabic System uses ten characters to express numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. The first and ninth inclusive are called digits, and the tenth is called zero, or cipher, or naught.

The word "digit" is derived from *digitus*, a finger, the fingers being the first natural counters. The first nine numbers are represented by nine digits, and the tenth is represented by the digit "1" and zero "0"; thus, 10. It is not known just when the symbol "0" was invented. It represents nothing when by itself.

The distinguishing feature of the Arabic System is its method of grouping numbers into tens and representing the *number* of groups by a digit and the *character* of the group by the *place* of the digit. The great value of the system comes of this feature.

This positional idea gives rise to two values for any one of the nine digits, — a *simple* value and a *local* value. The *simple* or *intrinsic* value of a figure is its value when alone. It is sometimes called the *absolute* value of a figure. The

local or *place* value of a figure is its value when taken in connection with another to express a number. Thus, in 245, the value expressed by "2" is two hundred; by "4," forty; and by "5," five. The simple value of "4" is *four*, the local value of it is *forty*.

The principle of local value, or the positional idea, is carried out to perfection because of the invention of "0." This idea is found in other systems to a limited degree; but since other systems have no "0," the idea cannot be so perfectly carried out in them. Should any of the series in a number be wanting, as in six hundred seven, where tens do not occur, the place is filled by "0" in the Arabic System, which throws the digit expressing hundreds in its proper place. Thus, 607.

THE RADIX

The *radix* in any system of numbers is the number of units which it takes of one order to make one of the next higher order. In the Arabic System the radix is ten, and is uniform. In the English System of weights and measures, the radix varies; for, twelve inches make one foot, three feet make one yard, five and one-half yards make one rod.

In any system the number of characters used, including 0, is the same as the number of units in the radix of that system. In the Common System this number is ten. A number system is named according to the number of units in the radix. Thus, the *binary* system has a radix *two*, the *quinary*, *five*.

In the *binary* system the radix is two, and the number of characters used is two. Any number may be expressed by two characters, or figures. These two are 0 and 1, and 1 stands for one, as it does in all systems. 10 stands for

two; 11 stands for three; 100 for four; 110 for six; 1111 for fifteen.

In the *ternary* scale the radix is three, and there are three characters used. They are 0, 1, and 2. In this. 10 stands for three; 11 for four; 20 for six; 221 for twenty-five. In the *quinary* scale the radix is five, and there are five characters used. They are 0, 1, 2, 3, 4. 10 stands for five; 30 for fifteen; 31 for sixteen; 321 for eighty-six. In the *octenary* scale the radix is eight, and there are eight characters used. They are 0, 1, 2, 3, 4, 5, 6, 7. 10 stands for eight; 21 for seventeen; 71 for fifty-seven; 721 for four hundred sixty-five.

In any scale the figure which expresses the greatest number expresses one less than the radix. The number 31 cannot occur in any scale below the *quaternary*. 541 cannot occur in any scale below the *senary*. 9 cannot occur in any scale below the decimal, or *denary*, since no scale below the denary uses the figure 9.

Reductions from one scale to another.

EXERCISES

1. Reduce 347 in the *quinary* to the *decimal* scale.

SOLUTION. The 3 represents $3 \times 5 \times 5 = 75$; the 4 represents $4 \times 5 = 20$; and the 7 represents 7. Hence, 347 in quinary represents $75 + 20 + 7 = 102$ in decimal. Using the subnotation, $347_5 = 102_{10}$.

2. Reduce to the decimal scale: 100_5; 124_5; 220_5; 742_8; 178_9; $101{,}021_3$; 4051_6; $12{,}131_4$.

3. Reduce 3702_{10} to the *senary* scale.

$$\begin{array}{r|l}
6 & 3702 \\ \hline
6 & \;\;617 - 0 \\ \hline
6 & \;\;102 - 5 \\ \hline
6 & \;\;\;\;17 - 0 \\ \hline
 & \;\;\;\;\;\;2 - 5
\end{array}$$

SOLUTION. In 3702 there are 617 6's and no units over; in 617 6's there are 102 6's of 6's with 5 6's over; in 102 6's of 6's there are 17 6's of 6's of 6's, or 17 216's, with no 36's over; in 17 216's there are 2 6's of 216's, or 2 1296's, with 5 216's over. $\therefore\ 3702_{10} = 25{,}050_6$.

4. Reduce 764_{10} to the binary scale; to the quinary; to the octenary; to the nonary.

5. Reduce 504_6 to the ternary scale; to the quinary; to the senary.

6. If θ is the next digit to 9, what is the radix of the system?

7. How many characters are there in the system in which $98\theta,621$ expresses a number? Name them.

NUMERATION AND NOTATION

The method of expressing numbers systematically by *words* is called *numeration;* and the method of expressing numbers systematically by symbols is called *notation.*

The common system of notation is further simplified by pointing off the figures into periods of three figures each. The first period is called units' period; the second to the left is called thousands' period; the next, millions' period; the next, billions' period.

Each period has units, tens, and hundreds of its kind. This is the custom of the French. The English plan is to group the figures into periods of six figures each. At least four periods of three figures each should be well learned, since that many are used in practice. Occasionally the fourth period is used, but not the fifth except in very rare instances. No one can appreciate a number expressed in the *third* period.

PRINCIPLES OF THE ARABIC SYSTEM

1. Each digit, in itself, always represents the same number of units. This has reference to the *simple value* of a figure.

2. The order of these units depends upon the place the digit occupies with reference to units' place. This has reference to the *local value* of a figure.

3. The sum of the values thus represented is the value of the whole expression.

4. Zero (0) has no value by itself, but is used to fill vacant places.

Numbers which are written in accordance with the Arabic Scale are called *Simple Numbers.* The idea of simple numbers may be so extended as to include all numbers written in a uniform scale, where the number of the characters used is the same as the number of units in the radix.

Numbers written in a scale which is not uniform are called *Compound Numbers.*

3 lb. 2 oz. 17 pwt. 13 gr. is a compound number, for the scale is not uniform. 17° 21′ 14″ is a compound number, for the radix is 60, and uniform, but the number of characters is not 60.

UNITY

The idea of *unity* underlies all number work, whether in Arithmetic or in Algebra.

Every number is derived from some fixed unit, either by multiplication or division; thus, the number five comes from taking something five times, and the number one-fifth comes from dividing something into five parts.

One's age is definitely expressed to another if some unit is agreed upon and number applied to it. Thus, if the year is the unit, and the number 19 is applied to it, the expression 19 years of age represents something definite. If the unit yard has the number 300 applied to it, the result 300 yards is a definite thing. But the same result is obtained if the unit 10 yards has the number 30 applied to it. The same result is obtained by applying the number 3 to the unit 100 yards, or by applying the number 600 to

the unit one-half yard, or by applying the number 1200 to the unit one-quarter yard.

Units are of two kinds: *primary* units and *derived* units. In pure numbers the unit "1" is the primary unit; and the units $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, 3, 8, and 13 are derived units. In turn, derived units are of two kinds: *multiple* units and *fractional* units. In denominate numbers the yard, the pound, the gallon, the dollar, are regarded as primary units; while the foot, the rod, the ounce, the quart, the barrel, the dime, the eagle, are derived units. Derived units are sometimes called *secondary* units.

If a unit is thought of as entire, complete, or as undivided, it is called an *integral* unit. The primary unit is always integral, while some derived units are integral and some are not. Hence, an *integral number* is one composed of integral units.

FUNDAMENTAL OPERATIONS

There are four and only four fundamental operations in number work: Addition, Subtraction, Multiplication, and Division.

ADDITION

Addition has its origin in counting. Addition is nothing more than abbreviated counting when the number to be added, or the addend, is 1. In passing from one number already formed to the next consecutive number the process is very simple. In passing from *four* to *five* the act is that of counting. In fact, the *beginner* passes from four to nine by counting the ones.

Addition is the process of combining two or more numbers into one number called their sum. The operation of addition is always possible, with such limitations only as are found in the following principles:

I. Only like numbers can be added. Thus, 4 apples and 3 apples are 7 apples. 4 apples cannot be added to 3 benches, and get a result that is alone apples or benches.

II. The sum is like the addends.

III. The sum is the same whatever the order of the addends. Thus, 4 pt. plus 3 pt. plus 2 pt. equals 9 pt.; also, 4 pt. plus 2 pt. plus 3 pt., or 3 pt. plus 2 pt. plus 4 pt., or 2 pt. plus 4 pt. plus 3 pt. equals 9 pt. This is usually called the Commutative Law of Addition.

IV. The numbers to be added may be grouped in any manner without affecting the sum. Thus,

$$
\begin{aligned}
(4\text{ lb.} + 5\text{ lb.} + 2\text{ lb.}) + 3\text{ lb.} &= (3\text{ lb.} + 5\text{ lb.} + 2\text{ lb.}) + 4\text{ lb.} \\
&= (2\text{ lb.} + 3\text{ lb.} + 4\text{ lb.}) + 5\text{ lb.}
\end{aligned}
$$

The sum is 14 lb. whatever the manner of grouping. This is called the Associative Law of Addition.

In preparing numbers for addition it is customary to write them so that units of the same kind shall stand in columns; but this must not be overdone, for it is often convenient to add numbers just as they are found.

Since numbers increase from right to left in place value, begin at the right to add, so that if the sum of the units of any order contains enough units to make one or more of a higher order, these higher units may be combined with the units of a like order.

As a check upon accuracy it is advisable to perform the addition a second time, not in the same, but in the reverse order.

Addition, after a fair amount of practice, seems to require little mental effort. Any suggestion, therefore, as to shortening the process, seems to be out of order. However, a few hints are here offered to save mental fatigue.

EXERCISE. Say four, six, eleven, fourteen, twenty-four; do not say four and two are six, six and five are eleven, eleven and three are fourteen, and so on. Take advantage of the commutative and associative laws, and change the order of the numbers in a column or group sets of numbers whose sum is ten. Thus, in the first column say four, fourteen, twenty-four. In the second, say ten, twenty, thirty, thirty-four. In the third, say three, thirteen, twenty-three, twenty-eight.

$$\begin{array}{r} 5,342 \\ 6,278 \\ 4,733 \\ 9,385 \\ 1,422 \\ 2,684 \\ \hline 29,844 \end{array}$$

In adding 494 and 296 it is easy to think four away from 494 and add it to 296; now say 490 and 300 = 790. This simplifies the process and so diminishes the likelihood of error.

$$685 + 565 = 700 + 550 = 1250.$$

$$2345 + 1205 = 2350 + 1200 = 3550.$$

The "700+550" and "2350+1200" are here put down to explain to the reader the mental acts; but in practice say at once, $685 + 565 = 1250$ and $2345 + 1205 = 3550$.

EXERCISES

1.	2.	3.	4.	5.
818	3943	3894	34,567	823,456
390	2846	295	89,012	92,134
970	8230	8672	34,567	5,678
273	7034	830	89,013	901
564	2875	29	45,678	23
983	5834	5678	90,234	4
756	2937	6789	56,789	198,765
745	2838	2005	1,234	893,256
307	7890	28	56,789	745,328
198	8326	943	12,345	943,456
209	9876	8649	67,890	823,921
896	6789	2083	13,579	678,932
24	9304	8256	24,680	468,246

6. Add 41,783, five million three hundred forty-two thousand six hundred seventy-two, $\overline{\text{M}}$DXCVIII, $\overline{\text{M}}$ $\overline{\text{DXLV}}$ DCXIV, and 349,764, and express the result in the Roman Notation.

7. The area of North America in square miles is 9,349,741 that of South America is 6,887,749; of Europe, 3,942,530; of Asia, 16,956,248; of Africa, 11,514,985; of other countries of the world combined, 3,709,781. What is the total land area of the globe?

8. The populations of the above countries are respectively: 88,005,695, 33,565,882, 360,580,788, 823,155,251, 168,498,091, and 5,683,968. What is the population of the entire globe?

9. In 1899 the acreage of wheat in the United States was 44,055,287; of corn, 77,721,781; of oats, 25,777,110; of rye, 1,643,207; of potatoes, 2,557,729; of barley, 2,583,125; of buckwheat, 678,332; and of hay, 42,780,827. Find the total acreage.

10. The above products were valued at $392,770,320, $552,023,428, $186,405,364, $11,875,350, $23,064,359, $5,271,462, $79,574,772, and $398,060,647 respectively. Find the total value of these products.

SUBTRACTION

Every mathematical operation has its inverse operation. The inverse operation undoes what the operation itself does.

Subtraction is the inverse of addition, and should be studied immediately after addition, for the relation between a process and its inverse is best seen when they are studied in close connection.

Numerical subtractions cannot always be made. Regard must be had to the numbers concerned. The subtrahend must never be greater than the minuend. Hence, the process of subtraction is a limited one.

Subtraction is the process of finding a number which, when added to the smaller of two numbers, will produce the larger. This operation can be performed by the simple act of counting. Independent of addition, subtraction is defined as the operation of finding the difference between two numbers. Subtraction is concerned with two ideas:

I. That of the difference between two numbers or quantities.

II. That of diminishing a given number or quantity. For example: A has a grammar class of 28 pupils, and B has a reading class of 21 pupils. The problem is to find the difference in the number of pupils in the two classes.

Again: A has a grammar class of 58 pupils, which is too large. A wishes to keep 32 of these pupils and with the

remainder form another class. How many will the other class contain?

In the first example the minuend and the subtrahend are two entirely distinct numbers or quantities, and the remainder is a part of the minuend. In the second example the minuend, subtrahend, and remainder are all found at first in one number or quantity. This second idea leads to another definition. Subtraction is the process of *diminishing* a number.

PRINCIPLES OF SUBTRACTION

I. Subtraction is concerned with like numbers only.

II. The remainder is like the subtrahend or minuend.

III. The minuend is equal to the sum of the subtrahend and the remainder.

IV. If the minuend and subtrahend are equally increased or diminished, the remainder is not affected. The difference in age between a parent and his child remains the same through life. If one box contains 75 pebbles and another contains 40 pebbles, the difference is 35 pebbles. This difference is not changed by putting 10 more pebbles in each box, nor by taking 30 pebbles out of each box. However, in this illustration, not more than forty pebbles could be taken from each box.

In preparing numbers for subtraction, follow the advice given in addition.

Since numbers increase from right to left in place value, for convenience begin at the right to subtract. Let the student state carefully the convenience.

ARITHMETIC COMPLEMENT

The difference between any number and a unit of the next higher order is called the *arithmetic complement* of

that number. Thus, 3 is the arithmetic complement of 7; 8 of 2; 11 of 89; 51 of 49; 120 of 880; 1001 of 8999.

THE OPERATION OF SUBTRACTION

Find the difference between 431,082 and 48,635.

431,082 . . .	Minuend.
48,635 . . .	Subtrahend.
382,447 . . .	Remainder.

5 units cannot be taken from 2 units; by Principle IV the same number may be added to the minuend and subtrahend without changing the remainder. In this case add 10 units to the minuend and its equal 1 ten to the subtrahend. Now 5 units from 12 units leaves 7 units, which write in the proper place. In the tens' column now 4 tens from 8 tens leaves 4 tens. Now add 10 hundreds to the minuend and its equal 1 thousand to the subtrahend; 6 hundreds from 10 hundreds leaves 4 hundreds. *Not* 8 thousands from 1 thousand, but 9 thousands from 1 thousand cannot be taken; hence add 10 thousands to the minuend and 1 ten-thousand to the subtrahend. 9 thousands from 11 thousands leaves 2 thousands. 5 ten-thousands from 43 ten-thousands leaves 38 ten-thousands.

This explains the somewhat prevalent practice of *adding* 1 to the next order in the subtrahend instead of *subtracting* 1 from the next order in the minuend, when a given order in the subtrahend contains more units than the corresponding order of the minuend.

EXERCISES

1. It is 1228 miles from Richmond, Ind., to Pueblo, Col.; from Richmond to St. Louis it is 308 miles (Penn. Lines); from St. Louis to Kansas City it is 285 miles. How far is it from Kansas City to Pueblo?

2. In the Old Testament there are 39 books, 929 chapters, 23,214 verses, 592,439 words, and 2,738,100 letters. In the New Testament there are 27 books, 260 chapters, 7950 verses, 182,253 words, and 933,380 letters. How many more of each kind are there in the Old than of the corresponding kind in the New Testament?

3. A carriage cost $145, and was sold at such a price that if $15 more had been received for it the gain would have been as much as the cost. Find the selling-price.

4. From $\overline{\text{M}}\ \overline{\text{CLV}}$ DXXXIV take 293,744.

5. If my cash balance in bank on a given morning was $2342.50, and I checked out during the day $314.24, $59.76, $82.75, $402.25, and $295.50, what was my balance for the next day?

6. If my balance in bank a given morning was $819.50, and I deposited during the day the following sums, $73.48, $64.19, $101.73, and $171, and checked out $214.50, $83.40, $501.10, and $21.32, find the balance at close of the day.

7. Two numbers differ by 314, and a third number is 1001, which is 182 more than the larger of the other two. Find the smallest number.

8. Abraham was born 2000 B.C., Moses, 1570 B.C., Thales, 640 B.C., Pythagoras, 580 B.C., Euclid, 350 B.C., and Washington, 1732 A.D. How many years have elapsed since the birth of each of these (1903 A.D.)?

MULTIPLICATION

Multiplication is a short method of addition when the numbers to be added are equal. This is a good definition so far as the fact is concerned, but it does not help the student in obtaining results.

Multiplication is the process of taking one number as many times as there are units in another. This definition is found often in text-books, but it will not hold except where the multiplier is a positive integer. Multiply 15 by $\frac{3}{4}$. There are 3 units in the multiplier, and 3 times 15 are 45, which is not the true product. Multiply 15 by -3. There are 3 units, each -1, and 3 times 15 is 45, which is not the true product.

Multiplication is the process of performing that operation upon one of two numbers, called the multiplicand, which is performed upon unity (1) to produce the other, called the multiplier. The number which is the result of the operation is called the product.

This definition is patterned after that of Charles Smith, and fulfils the requirement when the multiplier is a simple positive or negative number, when it is fractional, and when it is complex. It is an excellent one to use in the proofs of the three general problems in multiplication of fractions.

There are three ways of finding how many primary units there are in a certain number of times a given

number. The result may be found by counting, by addition, and by multiplication.

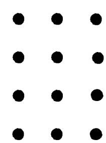
The operation of multiplication is always possible, except for the limitation implied under the first of the following principles:

PRINCIPLES OF MULTIPLICATION

I. The multiplier is always a pure number (abstract). Such an expression as 5 lb. multiplied by 3 ft. is absurd.

II. The product is always of the same kind as the multiplicand. This grows out of Principle II in addition.

III. The product of two numbers is the same whichever number is made the multiplier. $3 \times 4 = 4 \times 3$. The product is 12 in either case. 12 trees may be arranged in an orchard as is shown in the following diagram, which shows 3 rows of 4 trees in a row, or 4 rows of 3 trees in a row. This is known as the Commutative Law of Multiplication, and is generalized in $ab = ba$.



IV. If one of two numbers is multiplied by any number, and the other is divided by the same number, their product is not changed. Thus, $5 \times 8 = 40$. Now multiply 5 by 4 and get 20; and divide 8 by 4 and get 2. The product of 20 and 2 is 40, as is the product of 5 and 8.

V. If the multiplicand is multiplied by the several parts of the multiplier, the sum of the several products is the product of the two numbers.

1. $15 \times 8 = 120$. That *this* is so may be shown by addition:

$$15 + 15 + 15 + 15 + 15 + 15 + 15 + 15 = 120.$$

$8 = 5 + 3$. $15 \times 5 = 75$, and $15 \times 3 = 45$. $75 + 45 = 120$.
$8 = 6 + 2$. $15 \times 6 = 90$, and $15 \times 2 = 30$. $90 + 30 = 120$.

This is known as the Distributive Law of Multiplication.

2. $28 \times 14 = 392$. This may be shown true by addition:

$$14 = 10 + 4.$$

$$28 \times 10 = 280, \text{ and } 28 \times 4 = 112. \quad 280 + 112 = 392.$$

Now 15 could be multiplied by 8, and 28 by 14, at once without separating the multipliers into parts. But the value of Principle V is shown at once in trying to multiply 432 by 326. It would be difficult to multiply by 326 at once. But

$$326 = 300 + 20 + 6. \qquad 432 \times 300 = 129{,}600.$$
$$432 \times 20 = 8640. \qquad 432 \times 6 = 2592.$$
$$129{,}600 + 8640 + 2592 = 140{,}832.$$

To avoid writing 432, or the multiplicand, and the partial products so often, it is customary to use the following form:

(1)		(2)	
	432		432
	326		326
	2 592		2 592
	8 640		8 64
	129 600		129 6
	140,832		140,832

Since it is not necessary to write the zeros of 20 and 300, use form (2) instead of (1).

There are other ways of multiplying by numbers which are too large to handle at once, and their use should be encouraged because they are convenient.

$236 \times 45 = 10{,}620.$	$45 = 9 \times 5$ or $15 \times 3.$
$236 \times 9 = 2124.$	$2124 \times 5 = 10{,}620.$
$236 \times 15 = 3540.$	$3540 \times 3 = 10{,}620.$

This plan works to convenience only when the multiplier can be separated into factors which are small enough to avoid partial products.

Exercise. Multiply 465 by 24 in three ways, and state, with reasons, your choice for practice.

Since numbers increase from right to left in place value, for convenience begin at the right to multiply, so that if at any time a partial product should contain enough units to make units of a higher order, these units of a higher order may be combined with the units of that order in the next partial product.

DIVISION

This is the fourth and last of the fundamental operations of Arithmetic. It is also the most difficult, in that it implies all of the others. It is closely related to subtraction, being a short method of taking one number away from another several times.

Division is the *inverse* of multiplication; that is, it is the process of finding one of two numbers when their product and the other number are given. Hence, it should follow immediately after multiplication. This relation should never be lost from sight.

Division is twofold in its nature.

1. How many times are 6 pk. contained in 24 pk.?
2. Distribute 24 pk. among 6 people. How many will each receive?

From the first problem this definition follows:

Division is the process of finding how many times one number, or quantity, is contained in another of the same kind.

From the second problem this definition follows:

Division is the process of separating a given number, or quantity, into a given number of equal parts. Both of these definitions should be well learned.

Division is the process of performing that operation upon one of two numbers, called the *dividend*, which is

performed upon the other, called the ***divisor***, to obtain unity (1). The result of this operation is called the *quotient* of the two numbers. This is a most excellent definition with which to prove the three general problems in division of fractions, and "the inverting of the divisor."

The operation of division is not always possible. A number is not always contained in another, nor can it always be separated into equal integral parts. Additions and multiplications can always be performed. Subtractions and divisions cannot always be performed numerically. The fact that the latter are thus limited gives rise to other than *positive* and *integral* numbers. *Negative* and *fractional* numbers have their origin in impossible subtractions and divisions respectively.

PRINCIPLES

I. If the dividend and divisor are like numbers, the quotient is a pure number (abstract). If the divisor is a pure number (abstract), the quotient is like the dividend. In any instance the remainder is like the dividend.

II. If each of the several parts of the dividend be divided by the divisor, the sum of the partial quotients is equal to the true quotient. This is known as the Distributive Law of Division. Thus, $216 \div 18 = 12$. It is easily shown that 18 can be subtracted from 216 exactly 12 times. Hence, 12 is the true quotient. Now $216 = 126 + 54 + 36$. $126 \div 18 = 7$; $54 \div 18 = 3$; $36 \div 18 = 2$. $7 + 3 + 2 = 12$.

III. If the dividend and divisor be multiplied by the same number, or divided by the same number, the quotient remains unchanged. Thus, $48 \div 12 = 4$. Now divide

both 48 and 12 by 3 and the problem becomes $16 \div 4 = 4$, which result is the same as at first. Or, $36 \div 12 = 3$. Now multiply both 36 and 12 by 3 and then the problem is $108 \div 36 = 3$.

BEGIN AT THE LEFT TO DIVIDE

Since numbers decrease from left to right in place value, for convenience begin at the left to divide, so that if at any time there should be a remainder it may be reduced to a lower denomination and combined with the units of that denomination to form a new dividend. In very early times this new dividend was called a *dividual*.

SHORT DIVISION

Short division should precede long division. Great proficiency should be attained in short division with divisors from *one* to *twelve* inclusive. If this is practiced sufficiently, the habit of dividing such numbers as 24382 by 2 by long division will cease. There is less probability of error in short than in long division when the divisor is a small number.

When the divisor is too large to admit of short division at one operation, it is very frequently convenient to factor the divisor and then proceed by short division.

EXERCISES

1. Divide [illegible] by 225. 225 = [illegible].

[illegible]

This method is short and simple, and hence lessens the probability of error.

2. Divide 342,673 by 245. $245 = 5 \times 7 \times 7$.

$$\begin{array}{r|l}
5 & 342{,}673 \\ \hline
7 & 68{,}534 + 3 \text{ 1's.} \\ \hline
7 & 9{,}790 + 4 \text{ 5's, or 20.} \\ \hline
 & 1{,}398 + 4 \text{ 7's of 5 each, or 140.}
\end{array}$$

$$3 + 20 + 140 = 163.$$

$342{,}673 \div 245 = 1398$, with a remainder of 163.

3. Divide 2,956,471 by 6006. $6006 = 2 \times 3 \times 7 \times 11 \times 13$.

$$\begin{array}{r|l}
2 & 2{,}956{,}471 \\ \hline
3 & 1{,}478{,}235 + 1. \\ \hline
7 & \phantom{0{,}}492{,}745 + 0 \text{ 2's.} \\ \hline
11 & \phantom{0{,}0}70{,}392 + 1 \text{ 3 of 2 each, or 6.} \\ \hline
13 & \phantom{0{,}00}6{,}399 + 3 \text{ 7's of 6 each, or 126.} \\ \hline
 & \phantom{0{,}000{,}}492 + 3 \text{ 11's of 42 each, or 1386.}
\end{array}$$

$$1386 + 126 + 6 + 1 = 1519.$$

$2{,}956{,}471 \div 6006 = 492$, with a remainder of 1519. In this case it is shorter and easier to use long division at once.

4. Divide 15 bu. 3 pk. 3 qt. and 1 pt. by 126. $126 = 3 \times 6 \times 7$.

$$\begin{array}{r|cccc|l}
 & \text{Bu.} & \text{Pk.} & \text{Qt.} & \text{Pt.} & \\
3 & 15 & 3 & 3 & 1 & \\ \hline
6 & 5 & 1 & 1 & 0 & + 1 \text{ pt.} \\ \hline
7 & 0 & 3 & 4 & 0 & + 2 \text{ times 3 pt.} = 6 \text{ pt.} \\ \hline
 & 0 & 0 & 4 & 0 &
\end{array}$$

$$6 \text{ pt.} + 1 \text{ pt.} = 7 \text{ pt.}$$

15 bu. 3 pk. 3 qt. 1 pt. when divided among 126 persons gives 4 qt. to each, with 7 pt. remaining.

5. Divide 34,741 by 540 by short division.

6. Divide 80,002 by 429 by short division.

7. Divide 129,347 by 3575 by short division.

8. Divide 6 tons 14 cwt. 3 qr. 2 lb. 10 oz. by 165 by short division.

In the year 1570 there appeared:

> "Multiplication is mie vexation,
> And Division is quite as bad;
> The Golden Rule is mie Stumbling Stule,
> And Practice drives me mad."

CASTING OUT NINES

Every number may be regarded as equal to some multiple of 9 plus some remainder. Thus, $13 = 9 + 4$; $26 = 9 \times 2 + 8$; $61 = 9 \times 6 + 7$; $111 = 9 \times 12 + 3$; $321 = 9 \times 35 + 6$; $36 = 9 \times 4 + 0$; and $5 = 9 \times 0 + 5$.

The remainder found by dividing any number by 9 is called the *Excess of Nines* of that number. The *excess of nines* of any number is the same as the excess of 9's in the sum of the digits of that number. Thus, $574 = 9 \times 63 + 7$. The sum of the digits 5, 7, and 4 equals 16. $16 = 9 + 7$.

$1385 = 9 \times 153 + 8$. $1 + 3 + 8 + 5 = 17$. $17 = 9 + 8$.

This truth may be applied to test the accuracy of the result in any one of the four fundamental processes of Arithmetic.

ADDITION

The number of 9's in the sum of two or more numbers equals the sum of the 9's in them taken separately plus the number of 9's in the sum of the excesses.

Hence, the excess of 9's in the sum of two or more numbers equals the excess of 9's in the sum of the excesses.

29,342	. . .	2
84,083	. . .	5
8,756	. . .	8
72,345	. . .	3
93,124	. . .	1
287,650	. . .	19

The excesses of 9's in these numbers are respectively 2, 5, 8, 3, and 1, and their sum is 19. The excess of 9's in 19 is 1, and the excess in 287,650 is 1. The addition may be considered correct.

SUBTRACTION

Since subtraction is the inverse of addition, if the excess of the sum of the excesses of the subtrahend and remainder equals the excess of the minuend, the subtraction may be considered correct.

29,341	. . .	1
3,872	. . .	2
25,469	. . .	8

MULTIPLICATION

$325 \times 245 = 79{,}625.$

Multiplicand	325	. . .	1
Multiplier	245	. . .	2
	79,625	. . .	2

Since 325, or a multiple of 9 plus 1, is multiplied by 245, or a multiple of 9 plus 2, the product contains a multiple of 9 plus 1, first multiplied by a multiple of 9, and then multiplied by 2. Hence, the product contains a multiple of 9 plus the product of 1×2.

The excess of 9's in any product equals the excess in the product of the excesses of the multiplicand and multiplier.

When this is true the multiplication may be considered correct.

DIVISION

$185{,}504 \div 496 = 374.$

Dividend	185,504	. . .	5
Divisor	496	. . .	1
Quotient	374	. . .	5

Since division is the inverse of multiplication, the excess of the dividend equals the excess of the product of the excesses of the divisor and quotient. If there be a remainder, the excess of the dividend equals the excess of the product of the excesses of the divisor and quotient plus the excess of the remainder.

NOTE. The method by casting out 9's is not always a test of accuracy.

2,937	. . .	3
8,654	. . .	5
12,491	. . .	8

The excesses in these numbers are 3 and 5, and their sum is 8. 8 is also the excess in 12,491; but 12,491 is *not* the true sum of these numbers.

PRECEDENCE OF SIGNS

No notice need be taken of the order of additions and subtractions. When expressions contain all of the signs of the four fundamental operations those of multiplication and division must be performed before those of addition and subtraction, except when otherwise indicated.

EXERCISES

Simplify:

1. $5 + 3 - 4 - 2 + 7.$
2. $5 + 3 \times 5 - 13.$
3. $26 - 2 \times 5 - 4.$
4. $28 \div 7 + 6 - 3 \times 3 + 9.$
5. $15 \times 2 + 4 \div 2 - 5 \times 3.$
6. $675 \times 13 \div 25 + 6 \times 189 - 150 \div 30 + 181.$
7. $28 \times 6 + 96 \div 8 - 22.$
8. $52 + 39 \div 3 - 42 \times 3 + 6.$
9. $75 - 5 \times 3 + 3 \times 6 + 60 \div 4.$
10. $99 \times 8 + 17 - 3 \times 15 + 56 \div 28.$
11. $108 - 8 \times 12 - 98 \div 49 + 75 \div 15 - 10 \times 3.$

If braces, brackets, parentheses, or vinculums are used, the above law must be modified. Expressions in the braces, brackets, parentheses, and vinculums must be simplified first.

12. $(108 - 8) \times 12 - 98 \div 49 + 75 \div (15 - 10) \times 3.$
13. $105 \div 21 + 80 \div 5 \times 81 + 36 \div 9.$
14. $(105 \div 21) + (80 \div 5 \times 81) + (36 \div 9).$

15. $(105 \div 21 + 80 \div 5) \times (81 + 36 \div 9)$.

16. $(46 - 8) \times 11 + 17 \times 15 + 83 \times 4 - 327 + 10 - 32 \times 5$.

17. $(740 + 63 - 200) \div \overline{450 - 325 + 76} \times (65 - 35 + 10)$.

18. $47 + 1561 \div 7 + 142 \times 20 + 522 \div 87 + 207 \div 23 + 342$.

19. $9500 + 6200 - 9000 \div (3400 + 2100 - 5490)$
$\times \overline{6500 - 6400 + 200}$.

20. What number is that to which, if 16 be added, the sum multiplied by 8, and 13 be subtracted from the product, the remainder will be 339?

21. $(106{,}003 + 112{,}598) \div (337 + 698 + 800 + 208)$.

22. $649\frac{2}{3} + \frac{1}{3} \times 16.5 - 19\frac{1}{2} \div 6.5 + 17 \times (29\frac{1}{2} + 6\frac{2}{3})$.

EXERCISES

1. A horse and a carriage cost \$375, and the horse cost \$125 more than the carriage. Find the cost of each.

2. What number multiplied by 5 gives the same product as 24 multiplied by 10?

3. If a boy sold 3 pk. hickory nuts at 40¢ per peck, 11 lb. hazelnuts at 20¢ per pound, and 2 bu. walnuts at 75¢ per bushel, and with the money purchased a pair of shoes for \$1.50, a hat for \$2, 3 lb. candy at 20¢ per pound, and 4 pairs of hose, find the price of the hose per pair.

4. If a grocer buys 75 bu. potatoes at 20¢ per bushel, at what price per bushel must he sell them to realize a profit of \$10?

5. A grain dealer bought 300 bbl. pork for \$1400. He sold 100 bbl. at \$6.22 per barrel, 110 bbl. at \$6.80 per barrel, and 70 bbl. at \$7. The remainder spoiled, and his expenses were \$154. Find what he gained.

6. What is the issue of a paper in 52 weeks, published six days each week, each day's issue being 98,754 copies?

7. I bought 425 dozen of wine at 85 cents per bottle, and sold it so as to gain $ 112. For how much was the whole sold?

8. A merchant took to China 23,495 pieces of shirting of 28 yd. each, which he sold at 6¢ per yard; 96,750 yd. of flannel at 23¢ per yard, 37,490 handkerchiefs at 30¢ apiece, and 24,025 pairs of gloves at 42¢ per pair. He brought back 2465 chests of tea of 40 lb. each, which cost 22¢ per pound, 16,945 lb. raw silk at $ 6.40 per pound, and the remainder in cash. How much cash did he receive?

9. Multiply $\overline{\text{XV}}\text{DCXXV}$ by $\overline{\text{IX}}\text{CCXIII}$; divide the product by 625; from the quotient take the sum of 28,135 and CXIX.

FACTORS

A *factor* of a number is a maker of that number; or, a factor of a number is one of the numbers which, when multiplied together, will produce that number. Thus, 2 and 3 are factors of 6, and 5, 6, and 7 are factors of 210.

A prime number is a number that has no factors except itself and 1. A composite number is one that can be factored. 29 is a prime number and 28 is a composite number.

If a prime number is a factor of another number, it is a prime factor of that number. 7 is a prime factor of 28; while 4 is a factor of 28, it is not a prime factor of 28.

EXERCISES

1. Give all of the prime numbers from 1 to 100.

2. Give the prime factors of the composite numbers from 1 to 100.

3. Give the prime numbers from 1000 to 1100.

4. Give the prime factors of the composite numbers from 1000 to 1050.

5. When are two numbers prime to each other?

6. May two composite numbers be prime to each other? Illustrate.

DIVISORS

A *divisor* of a number is a factor of that number; or, it is a number which is contained in that number an integral number of times. The divisors of 45 are 3, 5, 9, 15, and 45.

EXERCISES

Give all of the divisors of 14; of 20; of 48; of 75; of 120; of 275; of 14,625; of 29,748.

PRINCIPLES

I. A divisor of a number is a divisor of any multiple of that number. Give proof.

II. A divisor of each of two numbers is a divisor of their sum. Give proof.

III. A divisor of each of two numbers is a divisor of their difference. Give proof.

A number may not be a divisor of either of two numbers, yet be a divisor of their sum, or of their difference. Thus, 5 is not a divisor of 26, nor of 4; yet it is a divisor of their sum, 30.

GREATEST COMMON DIVISOR

Give all of the divisors of 24; of 36. What divisors are common to 24 and 36? What is the greatest divisor that is common to 24 and 36?

The greatest divisor that is common to two or more numbers is called their *greatest common divisor.* The divisors of 30 are 2, 3, 5, 6, 10, 15, and 30. The divisors of 45 are 3, 5, 9, 15, and 45. The divisors 3, 5, and 15 are common to 30 and 45, and are called common divisors of

these two numbers. 15 is the greatest common divisor of 30 and 45. It will be observed that 15, the g. c. d., is the product of all of the *common prime* divisors. Hence, another definition of g. c. d. may be given: The g. c. d. of two or more numbers is the product of all of the common prime divisors or factors of these numbers. Verify this definition in other exercises. It is thought that the latter definition of g. c. d. is the better.

EXERCISES

1. Find the g. c. d. of 75, 120, and 135.

$$75 = 3 \cdot 5 \cdot 5$$
$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$$
$$135 = 3 \cdot 3 \cdot 3 \cdot 5$$

(*a*) SOLUTION. $\because$ 3 and 5 are the only common prime divisors, the g. c. d. equals $3 \times 5 = 15$.

(*b*) SOLUTION. All of the factors of 75 are 75, 25, 15, 5, and 3. The g. c. d. of 75, 120, and 135 cannot be greater than 75. If 75 is a divisor of each of the other numbers, it is the g. c. d. sought. 75 is not a divisor of 120; hence it need not be tried further, and is not the g. c. d. 15 is the next largest divisor of 75 which is a divisor of 120 and of 135. Hence 15 is the g. c. d. of 75, 120, and 135.

By this latter method it is not necessary to find the factors of any of the numbers given, except of the smallest of them. These two are called the "factoring method."

If the numbers are very large, either of the above methods is too laborious, for the factors may not easily be found by inspection. When this is the case, the g. c. d. should be found by the "long division method."

2. Find the g. c. d. of 5460 and 12,138. It is convenient to use the "long division method" in this problem. This method was used as long ago as 300 B.C.

SOLUTION

$$
\begin{array}{l}
5460)\,12138\,(2 \\
\quad\ \underline{10920} \\
\qquad 1218)\,5460\,(4 \\
\qquad\qquad \underline{4872} \\
\qquad\qquad\ 588)\,1218\,(2 \\
\qquad\qquad\qquad\ \underline{1176} \\
\qquad\qquad\qquad\quad 42)\,588\,(14 \\
\qquad\qquad\qquad\qquad\quad \underline{588}
\end{array}
$$

$\therefore$ 42 = the g. c. d. of these numbers.

EXPLANATION. $\because$ the g. c. d. of these numbers is a divisor of each, it $\ngtr$ 5460. If 5460 is a divisor of 12,138, it is the g. c. d. sought. By trial, 5460 is not a divisor of 12,138. $\therefore$ the g. c. d. $\neq$ 5460. $\because$ the g. c. d. is a divisor of 5460, it is a divisor of 10,920, Prin. I, and also of 1218, Prin. III. $\therefore$ the g. c. d. $\ngtr$ 1218. If 1218 is a divisor of 5460, it is the g. c. d. $\because$ 1218 is not a divisor of 5460, the g. c. d. $\neq$ 1218.

$\because$ the g. c. d is a divisor of 1218 and 5460, it is a divisor of 4872, Prin. I, and of 588, Prin. III.

$\therefore$ the g. c. d. $\ngtr$ 588. If 588 is a divisor of 1218, 5460, and 12,138, it is the g. c. d. 588 is not a divisor of 1218, hence the g. c. d. $\neq$ 588.

$\because$ the g. c. d. is a divisor of 588, it is a divisor of 1176, Prin. I, and of 42, Prin. III. $\therefore$ the g. c. d. $\ngtr$ 42. 42 is a divisor of 588, Prin. I; of 1218, Prin. II; of 4872, Prin. I; of 5460, Prin. II; of 10,920, Prin. III; and of 12,138, Prin. II.

$\because$ the g. c. d. $\ngtr$ 42, and 42 is a divisor of 5460 and 12,138, it is the g. c. d. sought.

The (*a*) solution by the "factoring method" is the best for general use in finding the g. c. d. of numbers.

THE LEAST COMMON MULTIPLE

A multiple of a number is a number which will contain it without a remainder.

4, 8, **12**, 16, 20, **24**, and 28 are multiples of 4.

6, **12**, 18, **24**, 30, 36, and 42 are multiples of 6.

12 and 24 are common multiples of 4 and of 6, while some of the multiples are not common.

12 is the least multiple that is common, and is called the least common multiple, l. c. m., of 4 and 6.

EXERCISES

1. Give a number of multiples of 12.

2. Give a number of multiples of 15.

3. Give some multiples of 12 which are also multiples of 15. What are these called?

4. What is the least multiple of 12 that is also a multiple of 15? What is it called?

5. Find the l. c. m. of 8, 12, and 15.

SOLUTION. The l. c. m. $= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$, as is evident by inspection. It is seen that 120 contains all of the different prime factors of the numbers, each factor being taken the greatest number of times it occurs in any one of the numbers.

$$8 = 2 \cdot 2 \cdot 2$$
$$12 = 2 \cdot 2 \cdot 3$$
$$15 = 3 \cdot 5$$

Hence the l. c. m. of two or more numbers is the product of all the different prime factors of the numbers, each factor being taken as many times as it occurs in any one of the numbers.

Whenever the numbers can be factored by inspection, it is advisable to find the l. c. m. by the "factoring method," as shown above.

When the numbers are large and cannot be factored by inspection, it is advisable to proceed as follows:

6. Find the l. c. m. of 5625, 24,128, and 13,624.

SOLUTION. The continued product of these numbers is *a* multiple of each. To see whether there is a smaller one, do as follows:

2	5,625	24,128	13,624
2	5,625	12,064	6,812
2	5,625	6,032	3,406
13	5,625	3,016	1,703
	5,625	232	131

The l. c. m. of these numbers equals $2^3 \cdot 13 \cdot 131 \cdot 232 \cdot 5625$.

Explanation. The prime factors of 232 are 2, 2, 2, and 29. 131 is prime. ∴ no other divisor is found in any two of the last three quotients, for 5625 does not contain 2, nor 29, nor 131.

The product of the divisors and quotients in the last line will be the product of all the different prime factors of the numbers, and hence must be the l. c. m. of the numbers by the definition of the l. c. m.

Remark. The product of the divisors above is $2 \cdot 2 \cdot 2 \cdot 13 = 104$. ∴ the true l. c. m. is $\frac{1}{104}$ of the continued product of the given numbers.

It can be shown that the above l. c. m. contains each of the given numbers, and that there is none smaller. Let the student do this.

7. Find the l. c. m. of 72, 96, and 120 by the "factoring method."

8. Find the l. c. m. of 368, 483, and 532 by the method in problem (6), usually called the "long division method."

EXERCISES

1. Find the g. c. d. and l. c. m. of 105, 231, and 1001.

2. Find the g. c. d. and l. c. m. of 936, 2925, and 225.

3. Find the g. c. d. and l. c. m. of 120, 228, and 720.

4. Show by example that the product of the g. c. d. and l. c. m. of two numbers equals the product of the numbers, and tell why this is so.

TESTS OF DIVISIBILITY FOR CERTAIN NUMBERS

Any number is divisible by 2 when the right-hand figure is 0 or represents a number divisible by 2; for, any number is some number of 10's plus the number represented by the right-hand figure. 10 is divisible by 2. Draw the conclusion.

Any number is divisible by 3 if the sum of its digits is divisible by 3. Thus, 477 is divisible by 3, since $4 + 7 + 7 = 18$, which is divisible by 3.

PROOF. In general, $477 = 4 \cdot \overline{10}^2 + 7 \cdot 10 + 7 = 4(3+7)^2 + 7(3 + 7) + 7 = 4(3^2 + 2 \cdot 3 \cdot 7 + 7^2) + 7(3 + 7) + 7 = 4 \cdot 3^2 + 4 \cdot 2 \cdot 3 \cdot 7 + 4 \cdot 7^2 + 7 \cdot 3 + 7^2 + 7$. Each term of the last expression which contains the factor 3 is divisible by 3. Hence, for our purpose, eliminate the terms $4 \cdot 3^2$, $4 \cdot 2 \cdot 3 \cdot 7$, and $7 \cdot 3$. The remaining terms are $4 \cdot 7^2$, 7^2, and 7. $4 \cdot 7^2 + 7^2 + 7 = 4(6 + 1)^2 + 7(6 + 1) + 7 = 4(6^2 + 2 \cdot 6 + 1) + 7(6 + 1) + 7 = 4 \cdot 6^2 + 4 \cdot 2 \cdot 6 + 4 + 7 \cdot 6 + 7 + 7$.

The terms containing 6 may be eliminated. The terms 4, 7, and 7 remain. $4 + 7 + 7 = 18$.

Now all the terms rejected contained 3, and 18 contains 3. Hence the general statement is true.

Any number is divisible by 4 whose two right-hand figures are 0's or represent a number which is divisible by 4; for, any number is some number of 100's plus the number represented by the two right-hand figures. 100 is divisible by 4. Draw the conclusion.

Any number is divisible by 5 if its right-hand figure is 0 or 5. Give proof.

Any number is divisible by 6 when it is divisible by 2 and by 3. Give proof.

Any number is divisible by 9 when the sum of its digits is divisible by 9. 531 is divisible by 9, since $5 + 3 + 1 = 9$.

PROOF. $531 = 5(9+1)^2 + 3(9+1) + 1 = 5(9^2 + 2 \cdot 9 + 1) + 3(9 + 1) + 1 = 5 \cdot 9^2 + 5 \cdot 2 \cdot 9 + 5 + 3 \cdot 9 + 3 + 1$. Reject all terms containing 9. $5 + 3 + 1$ remains. $5 + 3 + 1 = 9$. Draw the conclusion.

Any number is divisible by 11 when the sum of the numbers in the odd places minus the sum of the numbers in the even places is 0 or a multiple of 11. 2926 is divisible by 11, for $(6 + 9) - (2 + 2) = 11$.

PROOF. $2926 = 2(11-1)^3 + 9(11-1)^2 + 2(11-1) + 6 = 2(11^3 - 3 \cdot 11^2 + 3 \cdot 11 - 1) + 9(11^2 - 2 \cdot 11 + 1) + 2(11-1) + 6 = 2 \cdot 11^3 - 2 \cdot 3 \cdot 11^2 + 3 \cdot 11 \cdot 2 - 2 + 9 \cdot 11^2 - 9 \cdot 2 \cdot 11 + 9 + 2 \cdot 11 - 2 + 6$.

Reject those terms which contain 11. There remains $-2 + 9 - 2 + 6 = 11$. $\therefore$ 2926 is divisible by 11.

$6006 = 6(11-1)^3 + 6 = 6 \cdot 11^3 - 6 \cdot 3 \cdot 11^2 + 6 \cdot 3 \cdot 11 - 6 + 6$.

Reject the terms containing 11. There remains $-6 + 6 = 0$. $\therefore$ 6006 is divisible by 11.

EXERCISES

1. Show that 7,543,173 is divisible by 11, by applying the test.

2. Show that 20,000,002 is divisible by 11, by applying the test.

3. Prove, as above, that 96,745 is divisible by 11.

4. Show that 800,008 and 7,003,007 are divisible by 11, by applying the test.

5. Are 1,087,362; 987,654,321; 123,456,789 divisible by 9? Apply the test.

6. Are 783, 801, 20,202, 10,101, and 817,164 divisible by 3? Apply the test.

FRACTIONS

Integral numbers are natural numbers. Such numbers were devised to meet practical needs, and they have been in use since civilization began.

Fractional numbers, likewise, are natural numbers, and *they* were devised to meet practical needs quite as much as integral numbers. Fractional numbers are just as real as integral ones. The real needs of life demand both.

Negative, irrational, imaginary, and complex numbers "are creations of mathematicians, being devised to meet mathematical rather than practical needs." They are not used in the ordinary affairs of life.

The notion of a fractional number is not quite so simple as that of an integral number, because the unit is not so simple.

If anything is divided into two equal parts, each part is called one-half, and is expressed by $\frac{1}{2}$. If it is divided into three equal parts, each part is called one-third, and is represented by $\frac{1}{3}$; two of these parts are called two-thirds, represented by $\frac{2}{3}$. If it is divided into twenty-two equal parts, each part is called one twenty-second, represented by $\frac{1}{22}$; seven of these parts are represented by $\frac{7}{22}$, fifteen of them by $\frac{15}{22}$.

Each of several things of the same kind and size might be divided into eleven equal parts, whence three of them, twelve of them, fifteen of them, twenty-eight of them, or

eighty-one of them would be expressed as $\frac{3}{11}$, $\frac{12}{11}$, $\frac{15}{11}$, $\frac{28}{11}$, and $\frac{81}{11}$.

The half is not necessarily obtained by dividing the primary unit "1" into two equal parts. It may be obtained by dividing six into two equal parts, when three of these would be equal to one-half of all. \$$2\frac{1}{2}$ is one-half of \$5; 4 qt. is one-half of 8 qt.; $\frac{2}{3}$ is one-half of $\frac{4}{3}$.

$\frac{2}{3}$ lb. is not necessarily obtained by dividing 1 lb. into 3 parts and taking 2 of them; it may be obtained by taking $\frac{1}{3}$ of 2 lb. In either instance the result is 8 oz., whether $\frac{2}{3}$ of 1 lb. or $\frac{1}{3}$ of 2 lb. This twofold idea is plainly illustrated in \$$\frac{3}{4}$. Its meaning is not unique. \$$\frac{3}{4}$ may mean $\frac{3}{4}$ of \$1 or $\frac{1}{4}$ of \$3; in either case the result is 75 cents.

Such quantities as $\frac{2}{3}$ lb. troy, $\frac{4}{9}$ yd., \$$\frac{3}{5}$, and $\frac{3}{4}$ bu. are usually looked upon as fractional; yet they may be looked upon as integral; thus, 8 oz. troy, 16 in., 60 cts., and 3 pk. In thought, 15 oz. avoirdupois is fractional, as is $\frac{15}{16}$ lb. avoirdupois; in form, the one is integral, and the other is fractional.

EXERCISES

Express the following quantities in integral form:

1. $\frac{3}{4}$ pt.	4. $\frac{7}{12}$ lb. troy.	7. $\frac{6}{7}$ wk.	10. $\frac{7}{36}$ sq. ft.
2. $\frac{5}{8}$ lb. av.	5. $\frac{2}{3}$ yd.	8. \$$\frac{5}{10}$.	11. $\frac{8}{15}$ hr.
3. $\frac{3}{32}$ bu.	6. $\frac{12}{36}$ yd.	9. $\frac{8}{9}$ sq. yd.	12. $\frac{3}{12}$ da.

13. Name the primary unit in each of the quantities in Exercises 1 and 12 inclusive.

14. Name the derived unit in each in two ways.

15. Show by a diagram that $\frac{3}{4}$ of 1 ft. is the same in extent as $\frac{1}{4}$ of 3 ft.

16. That $\frac{3}{8}$ of 1 sq. ft. is equal to $\frac{1}{8}$ of 3 sq. ft.

17. Show by a drawing that $\frac{1}{9}$ yd. $= \frac{1}{3}$ ft. $= \frac{4}{12}$ ft. $=$ 4 in.

Definitions. A fractional number is one composed of fractional units. Thus, $\frac{6}{7}$ is a fractional number composed of 6 units each $\frac{1}{7}$, or 3 units each $\frac{2}{7}$, or 1 unit $\frac{6}{7}$.

In $\frac{6}{7}$, the 7 shows the number of parts into which the *primary* unit "1" has been divided, or it shows the number of parts into which the *derived* unit 6 has been divided. In either case it names the number of the parts into which the unit has been divided and is called the *denominator* of the fraction.

The number above the line is called the *numerator* of the fraction.

The numerator and the denominator are together called the *terms* of the fraction.

A fraction which has simple integral numbers for its terms is called a *simple* fraction.

If the numerator is smaller than the denominator, the fraction is called a *proper* fraction; and if the numerator is equal to, or greater than, the denominator, the fraction is called an *improper* fraction. The terms *proper* and *improper* are used for convenience only. In thought, $\frac{17}{5}$ is just as proper a fraction as is $\frac{4}{5}$.

In general, a fractional number is the result of an attempt to divide one number or quantity by another of the same kind. It was said on page 27, that there is a limit to the operation of division. The quotient is not always an integral number.

How often are 15 meters contained in 13 meters? Distribute 12 lb. sugar among 13 persons. In either case the result is fractional, and the statement "a fraction is an unexecuted division" seems to be true.

Hence, a fractional number is subject to all of the principles of division, regarding the numerator as the dividend and the denominator as the divisor.

PRINCIPLES

I. If the numerator of a fraction is multiplied by any number, the value of the fraction is multiplied by that number. Thus, $\frac{2 \times 3}{7} = \frac{6}{7}$. The size of the fractional unit has not been changed; hence, $\frac{6}{7}$ is three times as great as $\frac{2}{7}$.

II. If the denominator of a fraction is multiplied by any number, the value of the fraction is divided by that number. Thus, $\frac{2}{3 \times 4} = \frac{2}{12}$. The number of fractional units has not been changed, but the unit in $\frac{2}{12}$ is one-fourth as large as the unit in $\frac{2}{3}$. Hence, $\frac{2}{12}$ is one-fourth as great as $\frac{2}{3}$.

III. If both terms of a fraction are multiplied by any number, the value of the fraction is not changed. Thus, $\frac{3 \times 2}{5 \times 2} = \frac{6}{10}$. The unit in $\frac{3}{5}$ is two times the unit in $\frac{6}{10}$, but there are half as many of them.

IV. If the numerator is divided by any number, the value of the fraction is divided by that number. Thus, $\frac{6 \div 2}{7} = \frac{3}{7}$. The size of the unit is not changed, but there are half as many in $\frac{3}{7}$.

V. If the denominator is divided by any number, the value of the fraction is multiplied by that number. Thus, $\frac{7}{8 \div 2} = \frac{7}{4}$. The number of units is not changed, but the unit of the new fraction is twice as large as that of the original fraction.

VI. If both terms are divided by any number, the value of the fraction is not changed. Thus, $\frac{6 \div 3}{9 \div 3} = \frac{2}{3}$.

In the resultant fraction, the unit is three times as large as in the given one, but there are one-third as many.

EXERCISES

1. Make a drawing to show that Prin. I is true.

2. Make a drawing to show that Prin. III is true

3. Make a drawing to show that Prin. IV is true.

4. Make a drawing to show that Prin. VI is true.

5. Make a drawing to show that Prins. I and V are equivalent statements.

6. The same for Prins. II and IV.

EXERCISES

1. Show that $\frac{15}{24} = \frac{5}{8}$.

PROOF. Both terms of the fraction $\frac{15}{24}$ may be divided by 3, by Prin. VI. Performing this operation, the result is $\frac{5}{8}$.

SOLUTION. $\dfrac{15}{24} = \dfrac{15 \div 3}{24 \div 3} = \dfrac{5}{8}.$

2. Reduce to lower terms: $\frac{8}{12}$; $\frac{6}{18}$; $\frac{15}{45}$; $\frac{216}{270}$.

3. Reduce to lowest terms: $\frac{48}{60}$; $\frac{48}{72}$; $\frac{48}{80}$; $\frac{285}{760}$.

4. Reduce to lowest terms: $\frac{343}{405}$; $\frac{5625}{12680}$; $\frac{2001}{6483}$.

5. When a fraction is in its lowest terms, what is the relation between its terms?

6. If the terms of a fraction are not easily factored by inspection, reduce to lowest terms by dividing both by their g. c. d. Why will the fraction be in its lowest terms when so treated?

7. Reduce to lowest terms: $\frac{8484}{21210}$; $\frac{30045}{46069}$; $\frac{35536}{59967}$.

EXERCISES

1. Show that $\frac{3}{7} = \frac{9}{21}$.

PROOF. Both terms of $\frac{3}{7}$ may be multiplied by 3, by Prin. V. Performing this operation, the result is $\frac{9}{21}$.

SOLUTION. $\dfrac{3}{7} = \dfrac{3 \times 3}{7 \times 3} = \dfrac{9}{21}$. (Avoid $\frac{3}{7} \times \frac{3}{3} = \frac{9}{21}$.)

2. Reduce $\frac{3}{5}$ to 15ths; to 20ths; to 35ths; to 2335ths.

3. How is the factor by which each term is multiplied determined?

4. Reduce $\frac{7}{11}$ to 22ds; to 55ths; to 88ths; to 90ths.

SOLUTION. $90 \div 11 = 8\frac{2}{11}$. $\dfrac{7}{11} = \dfrac{7 \times 8\frac{2}{11}}{11 \times 8\frac{2}{11}} = \dfrac{57\frac{3}{11}}{90}$.

5. Reduce $\frac{4}{5}$ to 15ths; to 4375ths; to 36ths; to 4002ds.

6. Reduce $\frac{7}{8}$ to 16ths; to 24ths; to 48ths.

EXERCISES

1. Show that $3\frac{1}{2} = \frac{7}{2}$.

PROOF. In 1 there are $\frac{2}{2}$; in 3 there are $\frac{6}{2}$; $\frac{6}{2} + \frac{1}{2} = \frac{7}{2}$. $\therefore 3\frac{1}{2} = \frac{7}{2}$.

2. Give a short solution for the same result.

3. In $3\frac{1}{2}$ how many 4ths? 6ths? 14ths? 144ths?

4. In $17\frac{3}{16}$ how many 16ths? 48ths? 192ds? 512ths?

SOLUTIONS. (*a*) In 1 there are $\frac{512}{512}$. In 17 there are $\frac{8704}{512}$. In $\frac{3}{16}$ there are $\frac{96}{512}$. $\frac{8704}{512} + \frac{96}{512} = \frac{8800}{512}$. $\therefore 17\frac{3}{16} = \frac{8800}{512}$.

(*b*) Notice that $17\frac{3}{16} = \dfrac{275}{16}$. $\dfrac{275 \times 32}{16 \times 32} = \dfrac{8800}{512}$.

5. In $2\frac{1}{3}$ how many 3ds? 5ths? 20ths?

6. In 6 how many 4ths? 5ths? 19ths? 245ths?

EXERCISES

1. Show that $\frac{19}{3} = 6\frac{1}{3}$.

PROOF. In $\frac{3}{3}$ there is 1 primary unit. In $\frac{18}{3}$ there are 6 primary units. $\frac{19}{3} = \frac{18}{3} + \frac{1}{3} = 6 + \frac{1}{3} = 6\frac{1}{3}$. $\therefore \frac{19}{3} = 6\frac{1}{3}$.

2. Give a short solution for the same result.

3. How many primary units in $\frac{15}{4}$? $\frac{21}{6}$? $\frac{29}{7}$?

4. How many primary units in $\frac{310}{6}$? $\frac{758}{200}$? $\frac{810}{820}$?

5. How many primary units in $\frac{75}{3}$? $\frac{7}{3}$? $\frac{216}{80}$?

DEFINITIONS. Such numbers as $3\frac{1}{3}$ and $29\frac{4}{7}$ are called *mixed* fractions, being composed of two kinds of units, integral and fractional. Such expressions as $\frac{2}{3\frac{1}{3}}$, $\frac{7\frac{1}{2}}{3\frac{1}{4}}$, and $\frac{\frac{1}{3}}{\frac{4}{9}}$ merely indicate that certain operations are to be performed. They are formal expressions of quotients. For convenience they are called *complex* fractions.

There is no need of any name for such expressions as $\frac{1}{3}$ of $\frac{2}{5}$ of $\frac{6}{11}$, since the word "of" is nothing more than a symbol of multiplication.

Fractional numbers are subject to the four fundamental operations of Addition, Subtraction, Multiplication, and Division; and all of the laws governing these operations in integral numbers must obtain in fractional numbers.

EXERCISES IN ADDITION

1. Find the sum of 3 cakes and 5 cakes; of 3 bu. and 5 bu.; of 3 fourths and 5 fourths; of $\frac{3}{4}$ and $\frac{5}{4}$.

2. Find the sum of 7 lb. and 5 lb.; of 7 keys and 5 razors.

In the latter problem, no combination can be made so as to obtain either keys or razors, for keys and razors are not alike. No change can be made so as to make them alike.

3. Find the sum of $\frac{7}{3}$ and $\frac{5}{3}$; of $\frac{7}{4}$ and $\frac{5}{3}$.

The latter fractions are not alike, hence cannot be added without some change. Unlike the *keys* and *razors* in problem above, $\frac{7}{4}$ and $\frac{5}{3}$ can be made alike. This is done by Prin. III. $\frac{7}{4}=\frac{21}{12}$ and $\frac{5}{3}=\frac{20}{12}$. Hence $\frac{7}{4}+\frac{5}{3}=\frac{21}{12}+\frac{20}{12}=\frac{41}{12}=3\frac{5}{12}$. $\therefore\ \frac{7}{4}+\frac{5}{3}=3\frac{5}{12}$.

4. Find the sum of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{9}$, and $\frac{7}{12}$.

5. Find the sum of $17\frac{1}{2}$, $23\frac{1}{6}$, $9\frac{3}{4}$, and $203\frac{5}{6}$.

SOLUTION

$$17\frac{1}{2} + 23\frac{1}{6} + 9\frac{3}{4} + 203\frac{5}{6} = 17\frac{6}{12} + 23\frac{2}{12} + 9\frac{9}{12} + 203\frac{10}{12} = 252\frac{27}{12} = 254\frac{3}{12} = 254\frac{1}{4}.$$

(Do not change the integral portions to fractions.)

6. Find the sum of $\frac{2}{3}$, $4\frac{1}{2}$, $19\frac{7}{8}$, $102\frac{15}{16}$, and $2009\frac{7}{12}$.

7. Find the sum of $245\frac{63}{64}$, $1052\frac{15}{32}$, and $29,341\frac{29}{30}$.

EXERCISES IN SUBTRACTION

1. From 8 bu. take 5 bu.; from 8 thirds take 5 thirds; from $\frac{8}{3}$ take $\frac{5}{3}$.

2. From $\frac{8}{5}$ take $\frac{5}{6}$.

This cannot be done unless some changes are made. By Prin. III these fractions can be made alike. $\frac{8}{5} = \frac{48}{30}$ and $\frac{5}{6} = \frac{25}{30}$. Then $\frac{8}{5} - \frac{5}{6} = \frac{48}{30} - \frac{25}{30} = \frac{23}{30}$. $\therefore\ \frac{8}{5} - \frac{5}{6} = \frac{23}{30}$.

3. From $\frac{8}{9}$ take $\frac{3}{8}$; from $\frac{24}{25}$ take $\frac{36}{49}$.

4. From $4\frac{1}{2}$ take $2\frac{2}{3}$.

SOLUTION. $4\frac{1}{2} - 2\frac{2}{3} = 4\frac{3}{6} - 2\frac{4}{6}$. But $\frac{4}{6} > \frac{3}{6}$. $4\frac{3}{6} - 2\frac{4}{6} = 3\frac{9}{6} - 2\frac{4}{6} = 1\frac{5}{6}$.

Another form of solution:

$$\begin{array}{r} 4\frac{1}{2} = 4\frac{3}{6} = 3\frac{9}{6} \\ 2\frac{2}{3} = 2\frac{4}{6} = \underline{2\frac{4}{6}} \\ 1\frac{5}{6} \end{array}$$

Avoid this: $\begin{array}{l} 4\frac{1}{2} = \frac{3}{6} \\ 2\frac{2}{3} = \frac{4}{6} \end{array}$. This is a loose method, and involves statements that are not true.

5. From $203\frac{7}{8}$ take $19\frac{2}{3}$; from $1898\frac{29}{36}$ take $1187\frac{5}{12}$; from $18,934\frac{31}{60}$ take $12,672\frac{173}{180}$.

MULTIPLICATION OF FRACTIONS

There are three general problems in multiplication of fractions, and only three:

(1) To multiply a fraction by an integer. (2) To multiply an integer by a fraction. (3) To multiply a fraction by a fraction.

Illustration of Prob. (1): Find the cost of 8 yd. of cloth at $\$\frac{3}{4}$ per yard. Illustration of Prob. (2): Find the cost of $\frac{2}{5}$ of a bu. of corn at 60 cents per bu. Illustration of Prob. (3): Find the cost of $\frac{2}{3}$ lb. of candy at $\$\frac{2}{5}$ per lb.

I. $\frac{3}{5} \times 4 = \frac{12}{5}$. This is shown by a principle of multiplication which says the product is always like the multiplicand. 3 of any kind multiplied by 4 equals 12 of the same kind. 3 fifths $\times$ 4 = 12 fifths; or $\frac{3}{5} \times 4 = \frac{12}{5}$.

II. $7 \times \frac{2}{5} = \frac{14}{5}$. The Commutative Law says that $7 \times \frac{2}{5} = \frac{2}{5} \times 7$. Hence the problem turns at once into the type of Prob. I. $\frac{2}{5} \times 7 = \frac{14}{5}$.

III. $\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$. If the multiplier were 2, the product by Prob. I would be $\frac{4}{3}$. Thus, $\frac{2}{3} \times 2 = \frac{4}{3}$. The multiplier is $\frac{1}{5}$ of 2; hence the product is $\frac{1}{5}$ of $\frac{4}{3} = \frac{4}{15}$. See Prin. II, page 46. $\therefore \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$.

A better proof for all three of these problems is based upon the definition: To multiply one number by another is to perform that operation upon the first which being performed upon the unit "1" will produce the second.

I. $\frac{3}{5} \times 4 = \frac{12}{5}$. Proof: By the definition, 4 is produced by repeating 1 four times: hence $\frac{3}{5}$ must be repeated four times to produce the product.

$\frac{3}{5} + \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{12}{5}$. $\therefore \frac{3}{5} \times 4 = \frac{12}{5}$.

II. $7 \times \frac{2}{5} = \frac{14}{5}$. Proof: By the definition, $\frac{2}{5}$ is produced by repeating 1 two times and dividing the result by 5; hence repeat 7 two times and divide the result by 5. $7 \times 2 = 14$. $14 \div 5 = \frac{14}{5}$. $\therefore 7 \times \frac{2}{5} = \frac{14}{5}$.

III. $\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$. Proof: To produce $\frac{2}{5}$, 1 is repeated two times and the result divided by 5; hence repeat $\frac{2}{3}$ two times and divide the result by 5. $\frac{2}{3} \times 2 = \frac{4}{3}$. $\dfrac{4}{3 \times 5} = \frac{4}{15}$.
$\therefore \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$.

EXERCISES

By the latter method prove:

1. $\frac{8}{9} \times 4 = \frac{32}{9}$. 2. $19 \times \frac{2}{3} = \frac{38}{3}$. 3. $\frac{7}{8} \times \frac{15}{17} = \frac{105}{136}$.

Find the product of:

4. $8\frac{3}{11} \times 2\frac{3}{26}$.
5. $18\frac{2}{5} \times 8\frac{1}{4}$.
6. $10\frac{1}{2} \times 2\frac{5}{6} \times 3\frac{2}{20}$.
7. $2\frac{3}{4} \times 4\frac{4}{5} \times 5\frac{5}{6}$.
8. $\frac{11}{12} \times 3\frac{7}{9} \times \frac{14}{17} \times 3\frac{6}{7}$.
9. $168\frac{3}{4} \times 88\frac{1}{3}$.
10. $5\frac{6}{17} \times \frac{85}{144} \times 1\frac{1}{18} \times \frac{36}{49}$.
11. $2\frac{1}{8} \times 1\frac{1}{84} \times 1\frac{4}{17} \times \frac{8}{85}$.
12. $\frac{2}{3}$ of $\frac{25}{28}$ of $7\frac{1}{5} \times \frac{7}{10} \times 16\frac{1}{60}$.
13. $29503\frac{2}{3} \times 10802\frac{5}{7}$.
14. What is the product of $\frac{7}{11}$ of 15, $\frac{14}{15}$ of $11\frac{5}{8}$, and $\frac{11}{31}$ of $\frac{3}{5}$?

DIVISION OF FRACTIONS

All of the various problems which may come up in *division of fractions* may be reduced to one or another of three general problems: (1) To divide a fraction by an integer. (2) To divide an integer by a fraction. (3) To divide a fraction by a fraction.

Illustrations: (1) If 4 yd. of cloth cost \$$\frac{8}{10}$, find the cost of 1 yd. (2) If 1 yd. of cloth cost \$$\frac{4}{5}$, find the number of yards that can be bought for \$8. (3) If 1 yd. of cloth cost \$$\frac{3}{10}$, how many yards can be bought for \$$\frac{4}{5}$?

I. $\frac{8}{11} \div 4 = \frac{2}{11}$. Proof: Since division is the process of performing that operation upon the dividend which must be performed upon the divisor to produce 1, and 4, the divisor, may be multiplied by $\frac{1}{4}$ to produce 1; hence $\frac{8}{11}$, the dividend, must be multiplied by $\frac{1}{4}$ to produce the true quotient. $\frac{1}{4}$ of $\frac{8}{11} = \frac{2}{11}$. $\therefore \frac{8}{11} \div 4 = \frac{2}{11}$.

II. $5 \div \frac{2}{3} = \frac{15}{2}$. Proof: Since $\frac{2}{3}$ must be multiplied by $\frac{3}{2}$ to produce 1, we must multiply 5 by $\frac{3}{2}$ to get the true quotient. $5 \times \frac{3}{2} = \frac{15}{2}$. $\therefore 5 \div \frac{2}{3} = \frac{15}{2}$.

III. $\frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$. Proof: Since $\frac{3}{4}$ must be multiplied by $\frac{4}{3}$ to produce 1, we must multiply $\frac{2}{5}$ by $\frac{4}{3}$ to produce the true quotient. $\frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$. $\therefore \frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$.

In each of the three proofs notice that the divisor has been *inverted*, and then used as multiplier of the dividend to produce the true quotient. Hence, in division of fractions, "invert the divisor and multiply" to produce the true quotient.

EXERCISES. Basing your proof on the above definition, prove the following results:

(1) $\frac{3}{7} \div 2 = \frac{3}{14}$. (2) $8 \div \frac{3}{5} = \frac{40}{3}$. (3) $\frac{2}{9} \div \frac{3}{5} = \frac{10}{27}$.

No doubt the easiest way to prove the results in division of fractions is to base the proof on the following definition of division, "Division is the process of dividing one number by another of the same kind."

I. $\frac{3}{5} \div 4 = \frac{3}{20}$. Proof: $\frac{3}{5} \div 4 = \frac{3}{5} \div \frac{20}{5}$. 3 of any kind divided by 20 of the same kind equals $\frac{3}{20}$. $\therefore \frac{3}{5} \div 4 = \frac{3}{20}$.

II. $9 \div \frac{4}{5} = \frac{45}{4}$. Proof: $9 \div \frac{4}{5} = \frac{45}{5} \div \frac{4}{5} = 45$ fifths $\div$ 4 fifths $= \frac{45}{4} = 11\frac{1}{4}$. $\therefore 9 \div \frac{4}{5} = 11\frac{1}{4}$.

III. $\frac{7}{9} \div \frac{3}{2} = \frac{14}{27}$. Proof: $\frac{7}{9} \div \frac{3}{2} = \frac{14}{18} \div \frac{27}{18}$. 14 things divided by 27 of the same kind equals $\frac{14}{27}$. $\therefore \frac{7}{9} \div \frac{3}{2} = \frac{14}{27}$.

In each problem the same result would have been obtained if the divisor had been inverted and used as a multiplier of the dividend. Hence to "invert the divisor and multiply" is equivalent to reducing the fractions, or dividend and divisor, to the same denomination, and finding the quotient of the numerators.

EXERCISES. By this last definition of division prove:

(1) $\frac{11}{12} \div 3 = \frac{11}{36}$. (2) $13 \div \frac{5}{6} = \frac{78}{5}$. (3) $\frac{11}{13} \div \frac{7}{8} = \frac{88}{91}$. (4) Verify the statement, "invert the divisor and multiply," in (1), (2), and (3).

Another definition of division: Division is the process of finding one of two numbers when their product and the other number are given.

By this definition prove:

(1) $\frac{2}{5} \div 3 = \frac{2}{15}$. (2) $4 \div \frac{3}{5} = \frac{20}{3}$. (3) $\frac{2}{7} \div \frac{3}{2} = \frac{4}{21}$.

EXERCISES

1. A cake is divided into two equal parts. What is each part called?

2. A year's salary is divided into three equal parts. What is one part called? What are two parts called? What are the three parts called?

3. A post is driven so that one-third of it is in the ground, and the part above the ground is 12 ft. long. Find the length of the post.

4. A post is one-half in the ground and 5 ft. above the ground. Find its length.

5. The half of an estate of $1200 is divided equally among 4 children. Find each one's share.

6. If I sold $\frac{3}{4}$ of a dozen bottles of vinegar, how many bottles were left?

7. A post is driven through the water into the ground; $\frac{1}{4}$ of it is in the ground; $\frac{1}{2}$ of it is in the water; and 12 ft. is above the water. What is the length of the whole post? What length is in the ground? What length is in the water?

8. Distribute $\frac{7}{12}$ of a pound of gold dust among 7 persons. State what each person gets both as a fractional and as an integral quantity.

9. Divide $\frac{5}{14}$ of 7 pounds of butter among 4 persons.

10. Divide $\frac{3}{8}$ of 12 yd. of ribbon among 18 persons.

11. If my salary is \$1200 for 1 year, what is it for 7 months?

12. An estate was divided equally among 9 children; 5 sons together received \$1080. How much did the 4 sisters together receive?

13. After expending $\frac{3}{4}$ of my money I had \$15 left; how much had I at first?

14. $\frac{2}{3} + 4\frac{7}{11} + 6\frac{7}{8} - \frac{2}{9} - 1\frac{17}{22} - 2\frac{2}{5} = ?$

15. $27\frac{96}{100} - 22\frac{59}{100} + 17\frac{22}{100} + 11\frac{85}{100} - 2\frac{2}{5} = ?$

16. $\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \frac{7}{8} + \frac{8}{9} - \frac{9}{10} = ?$

17. Which is the greatest: $\frac{3}{4}$, $\frac{90}{121}$, or $\frac{45}{59}$?

18. What quantity is $> 29\frac{6}{7}$ by $19\frac{2}{3}$?*

19. From what number must $14\frac{2}{7}$ be subtracted to leave $\frac{1}{7} + \frac{3}{4}$?

20. What number must be added to $\frac{2}{3}$ of $13\frac{1}{2}$ to make $15\frac{3}{4}$?

21. From what number must $7\frac{3}{8}$ be subtracted to leave $\frac{2}{7}$ of $21\frac{7}{8} - 1\frac{1}{4}$?

22. The sum of two numbers is $\frac{11}{23}$ of $\frac{92}{77}$ and one of them is $\frac{2}{9}$; find the other.

23. Explain the reduction of $7\frac{2}{9}$ to $\frac{65}{9}$.

24. Explain the reduction of $\frac{181}{7}$ to $25\frac{6}{7}$.

25. Add $\frac{249}{260}$, $\frac{217}{340}$, $\frac{21}{361}$, and $\frac{1723}{4199}$.

26. $\frac{1}{9}$ of $\frac{81}{85}$ of $\frac{17}{24}$ of $\frac{31}{34}$ of $\frac{51}{93} = ?$

27. Divide $\frac{19}{87}$ of $\frac{58}{76}$ by $\frac{13}{20}$ of $\frac{75}{91}$.

28. Is the symbol correctly placed in $\frac{18}{17} > \frac{35}{33}$?

29. If a man's pace is $2\frac{1}{4}$ ft., how many paces will he take in going $3\frac{3}{8}$ miles?

* The sign $>$ is read "greater than"; thus, $6 > 4$. The sign $<$ is read "less than"; thus, $5 < 8$.

30. 5 da. 6 hr. 15 min. $\div 2\frac{1}{3} = ?$

31. $3\frac{1}{2}$ of $\frac{8}{9} \times \frac{4}{7} \div 2\frac{2}{3} = ?$

32. The difference between a certain number and $5 \div 1\frac{2}{3} = 8\frac{3}{4} \div 7$. Find the certain number.

33. What fraction repeated 7 times, and this result increased by $21\frac{2}{9}$, is equal to $100 - 13 \div \frac{1}{6}$?

34. What fractional number divided by $12\frac{1}{2}$ will yield $12\frac{1}{2}$?

35. From what number can $3\frac{19}{60}$ be taken 7 times without any remainder?

36. From what fraction can $7\frac{2}{3}$ be taken $8\frac{1}{7}$ times and leave a remainder of $18\frac{5}{42}$?

37. What is the effect on the fraction $\frac{8}{11}$ if 3 is added to both terms?

38. What is the effect on the value of $\frac{11}{8}$ if 3 is added to both terms?

39. Make a general statement as to the effect of adding an integer to both terms of a fraction.

40. Investigate the effect of adding a proper fraction to both terms of a fraction.

41. Investigate the effect of subtracting an integer, or a proper fraction, from both terms.

42. Simplify $\dfrac{2\frac{1}{2}}{5\frac{2}{3}}$.

SOLUTION. $\dfrac{2\frac{1}{2}}{5\frac{2}{3}} = \dfrac{2\frac{1}{2} \times 6}{5\frac{2}{3} \times 6} = \frac{15}{34}$.

43. Simplify $\dfrac{18\frac{2}{7}}{\frac{2}{3} \text{ of } \frac{3}{5} \text{ of } \frac{5}{2}}$.

44. Simplify $\dfrac{3\frac{1}{3} - 2\frac{1}{5}}{9\frac{2}{7} + 5\frac{1}{4}}$.

SOLUTION. $\dfrac{3\frac{1}{3} - 2\frac{1}{5}}{9\frac{2}{7} + 5\frac{1}{4}} = \dfrac{3\frac{5}{15} - 2\frac{3}{15}}{9\frac{8}{28} + 5\frac{7}{28}} = \dfrac{1\frac{2}{15}}{14\frac{15}{28}} = \frac{17}{15} \times \frac{28}{407} = \frac{476}{6105}$.

45. $\dfrac{\frac{3}{8} \text{ of } \frac{16}{17}}{5\frac{5}{6} - 2\frac{11}{24}} \div \dfrac{1\frac{7}{9}}{12\frac{3}{4}} = ?$

46. $\dfrac{8\frac{1}{2} \times \frac{7}{13}}{\frac{1}{12} \text{ of } \dfrac{3\frac{1}{2}}{\frac{4}{5}}} \div \dfrac{\frac{1}{3}}{\frac{3}{4} \text{ of } \frac{7}{11}} = ?$

47. Divide $\frac{5}{9} \times \dfrac{3\frac{2}{3}}{\frac{5}{6} \times 1\frac{2}{9}}$ by $\dfrac{7\frac{2}{3} - 4\frac{1}{2}}{\frac{1}{14} \times 3\frac{5}{6}} \times \dfrac{7\frac{2}{3}}{31\frac{8}{9}}$.

48. $\frac{3}{7} \text{ of } \dfrac{4\frac{5}{9}}{12\frac{1}{3}} \text{ of } \dfrac{3\frac{4}{11}}{11\frac{5}{7}} \div 1\frac{1}{11} = ?$

49. The cargo of a vessel is worth \$10,000. $\frac{4}{9}$ of $\frac{5}{7}$ of $\frac{9}{10}$ of the vessel is worth $\frac{1}{4}$ of $\frac{4}{5}$ of $1\frac{5}{7}$ of the cargo. Find the value of the vessel and the cargo.

50. A merchant sold 5 bbl. apples for \$$32\frac{1}{2}$, which was $\frac{5}{8}$ as much as he received for all he had left at \$4 per bbl. How many bbl. in all did he sell?

51. If a certain number is increased by $1\frac{3}{4}$, this sum diminished by $\frac{3}{8}$, this remainder multiplied by $5\frac{3}{5}$, and this product divided by $1\frac{2}{7}$, the quotient is $7\frac{1}{2}$. Find the number.

THE G. C. D. AND L. C. M. OF FRACTIONS

The g. c. d. of 6 bu., 3 bu., and 9 bu. is 3 bu. by the definition of g. c. d. The g. c. d. of $\frac{6}{7}$, $\frac{3}{7}$, and $\frac{9}{7}$ is $\frac{3}{7}$ by the same definition.

The g. c. d. of 12 qt., 2 gal., and 4 pt. is easily found, if expressed in terms of the same unit — pint. When so expressed the problem is, find the g. c. d. of 24 pt., 16 pt., and 4 pt. By definition the g. c. d. is 4 pt. Find the g. c. d. of $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{6}$. When reduced to the same denomination these fractions become $\frac{20}{30}$, $\frac{18}{30}$, and $\frac{25}{30}$. The g. c. d. is evidently $\frac{1}{30}$.

$\frac{20}{30} = \frac{1}{30} \times 2 \times 2 \times 5.$
$\frac{18}{30} = \frac{1}{30} \times 2 \times 3 \times 3.$
$\frac{25}{30} = \frac{1}{30} \times 5 \times 5.$

No divisor is common to the three fractions except $\frac{1}{30}$, which is therefore the g. c. d. by def. of g. c. d.

The l. c. m. of 6 bu., 3 bu., and 9 bu. is 18 bu. by the definition of the l. c. m. The l. c. m. of $\frac{6}{7}$, $\frac{3}{7}$, and $\frac{9}{7}$ is $\frac{18}{7}$ by the same definition.

The l. c. m. of 12 qt., 2 gal., and 4 pt. can be found easily by reducing them to the same unit — pint. Then the problem is: Find the l. c. m. of 24 pt., 16 pt., and 4 pt.; which is 48 pt. by definition.

The l. c. m. of $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{6}$ is found by reducing these fractions to the same denomination, when they become $\frac{20}{30}$, $\frac{18}{30}$, and $\frac{25}{30}$. It is evident that the l. c. m. is $\frac{900}{30}$, or 30.

$\frac{20}{30} = \frac{1}{30} \times 2 \times 2 \times 5.$
$\frac{18}{30} = \frac{1}{30} \times 2 \times 3 \times 3.$
$\frac{25}{30} = \frac{1}{30} \times 5 \times 5.$

The l. c. m. $= \frac{1}{30} \times 2 \times 2 \times 3 \times 3 \times 5 \times 5$, which is equal to $\frac{900}{30} = \frac{30}{1} = 30.$

Observe that the g. c. d., $\frac{1}{30}$, has for its numerator the g. c. d. of the numerators of the fractions as given at first, and for its denominator the l. c. m. of the denominators of the given fractions.

Observe also that the l. c. m., $\frac{30}{1}$, has for its numerator the l. c. m. of the numerators of the given fractions, and for its denominator the g. c. d. of the denominators of the given fractions.

EXERCISES

1. Find the g. c. d. of $\frac{3}{5}$, $\frac{6}{7}$, $\frac{9}{14}$, and $\frac{12}{35}$.
2. Find the g. c. d. of $15\frac{2}{3}$, $8\frac{3}{5}$, and $21\frac{1}{6}$.
3. Find the l. c. m. of $9\frac{3}{5}$, $6\frac{3}{10}$, and $1\frac{1}{2}$.

NOTE. All fractions must be in their lowest terms.

DECIMAL FRACTIONS

Recall the definitions of an integral unit, an integral number, a fractional unit, and a fractional number.

A decimal fractional unit is one of the decimal divisions of the primary unit 1. Thus, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and $\frac{1}{10000}$ are such units.

Hence, a decimal fraction is one composed of decimal fractional units. Thus, $\frac{3}{10}$, $\frac{173}{100}$, $\frac{29}{1000}$, and $\frac{7}{10000}$ are decimal fractions.

A decimal fraction is well defined by saying that it is a fraction whose denominator is 10 or some *power* of 10.

It is customary to express a decimal fraction by the use of the decimal point. $\frac{3}{10} = .3$; $\frac{173}{100} = 1.73$; $\frac{29}{1000} = .029$; and $\frac{7}{10000} = .0007$.

The decimal point was invented in the seventeenth century A.D. Decimal fractions were in use probably 3000 years before the decimal point was invented. Hence, it is *not* essential that a decimal fraction be expressed thus: .32. $\frac{32}{100}$ is just as truly a decimal fraction as .32.

All fractions whose units are *not* decimal divisions of the unit 1, or whose denominators are *not* some positive integral power of 10, are called *common* fractions. Most English text-books call them *vulgar* fractions.

29.731 is read 29 and 731 thousandths; 2.4 is read 2 and 4 tenths, 24 tenths, or 240 hundredths, or .24 tens, etc.; 3.15 is read as 3 and 15 hundredths, 31.5 tenths, or 315 hundredths, 3150 thousandths, etc.

PRINCIPLES

I. A decimal fraction, expressed in the usual form, may be multiplied by 10, 100, 1000, or 10,000 by moving the decimal point one, two, three, or four places to the right. Thus, $.26 \times 10 = 2.6$; $.26 \times 100 = 26$; $.26 \times 1000 = 260$; $.26 \times 10{,}000 = 2600$.

II. A decimal fraction may be divided by 10, 100, 1000, or 10,000 by moving the decimal point one, two, three, or four places to the left. Thus, $2.34 \div 10 = .234$; $2.34 \div 100 = .0234$; $2.34 \div 1000 = .00234$.

What principles of fractions in general are here at issue in these principles?

Addition and subtraction of decimals offer no difficulties, except such as are met in simple numbers.

MULTIPLICATION OF DECIMALS

There are three general problems in multiplication of decimal fractions, as follows:

I. To multiply a decimal fraction by an integer.
II. To multiply an integer by a decimal fraction.
III. To multiply a decimal fraction by a decimal fraction.

1. $.37 \times 4 = 1.48$.

PROOF. 37 hundredths $\times 4 = 148$ hundredths or 1.48, by the principle that the product is always like the multiplicand.

2. $12 \times .4 = 4.8$.

PROOF. By Prins. I and II of decimal fractions the problem may be made to assume the following form without affecting the product: $12 \times .4 = 1.2 \times 4 = 4.8$, by

Prob. 1. Or, by the Commutative Law, $12 \times .4 = .4 \times 12 = 4.8$, by Prob. 1.

3. $.12 \times .4 = .048$.

PROOF. To multiply the multiplier by 10 and divide the multiplicand by 10 will not affect the product, whence $.12 \times .4 = .012 \times 4 = .048$, by Prob. 1.

These three general problems may be proven by the definition of multiplication: It is the process of performing that operation upon the multiplicand which must be performed upon unity "1" to produce the multiplier.

I. $.12 \times 4 = .48$. Unity "1" must be repeated 4 times to produce 4; hence .12 must be repeated 4 times to produce the product, which is done by the laws of addition. $.12 + .12 + .12 + .12 = .48$. Hence, $.12 \times 4 = .48$.

II. $12 \times .4 = 4.8$. Unity "1" must be divided by 10 and the result multiplied by 4 to produce .4; hence 12 must be divided by 10, which gives 1.2 by Prin. I of decimal fractions, and $1.2 \times 4 = .48$, by Prob. I.

III. $.12 \times .4 = .048$. The process is the same as in II. Repeat it, and show the result.

After making observations in the above explanations, state a rule for pointing off decimal places in the product.

DIVISION OF DECIMALS

I. To divide a decimal fraction by an integer.

II. To divide an integer by a decimal fraction.

III. To divide a decimal fraction by a decimal fraction.

These are the three general problems in division of decimals.

I. $.126 \div 6 = .021$; for 126 of any kind divided by 6 equals 21 of the same kind.

II. $65 \div .5 = 130$. **Both dividend and divisor may be multiplied by 10 without changing the quotient. Hence,** $65 \div .5 = 650 \div 5 = 130$.

III. $.248 \div .8 = .31$, **for** $.248 \div .8 = 2.48 \div 8 = .31$, **as explained in II.**

II and III are nicely proved by the definition: Division is the process of performing upon the dividend that operation which must be performed upon the divisor to produce 1.

In II, since .5 multiplied by 2 gives 1, 65 multiplied by 2 will give the true quotient. $65 \times 2 = 130$.

In III, since .8 multiplied by $1\frac{1}{4}$ **gives 1, .248 multiplied by** $1\frac{1}{4}$ **gives the true quotient.** $.248 \times 1\frac{1}{4} = .31$.

EXERCISES

1. $181 \times 1.6 = ?$ $2876 \times .003 = ?$ $20{,}006 \times .75 = ?$

2. $2.965 \times 27 = ?$ $.0061 \times 24 = ?$ $12.96 \times 7 = ?$

3. $2.83 \times .17 = ?$ $.817 \times .02 = ?$ $167.3 \times .2114 = ?$

4. Show that adding a 0 to the right of a decimal fraction does not change its value. Thus, $.24 = .240$.

5. What is the effect of placing a 0 between the point and the 6 in .6? Explain fully.

6. $.296 \div 8 = ?$ $56.25 \div 25 = ?$ $9.87 \div 7 = ?$

7. $.875 \div .25 = ?$ $2001 \div .03 = ?$ $218{,}097 \div .03 = ?$

8. $3.006 \div .06 = ?$ $81.909 \div .9 = ?$ $156.25 \div .25 = ?$

9. $6.003 \times .4 \div .12 = ?$

10. $.75 + .25 \times 3 + 1.4 \div .7 - .32 \times 4 = ?$

EXERCISES

Find the value of:

1. $8.763 - 4.12 + 82.756 + 2.1134 - 63.056$.

2. $196.73 + 24.2913 - 98.217 - 45.64 + 7.6$.

3. $6.45 \times 4.4 + 219 \times .005 - 7.5 \times 4.8 - .64 \times 2.17$.

4. $4.75 \times 8 + 7.248 \div 120 + 8.56 \times .07 + .071 \times 36$.

5. $96 \times 1.12 - (75 - 1.6 \times 2.5) + 96 \times (5 \times .0108 + .3642)$.

6. Divide the sum of 16 thousandths and 16 ten-millionths by their difference.

7. Multiply 64 ten-thousandths by 5 and 2 thousandths, and divide the result by 2 millionths.

8. What number divided by 32.75 will give 24.072 with a remainder of 1.3465?

9. How many rods of fence are necessary to inclose a field (rectangular) 75.364 rods long and 45.216 rods wide?

10. The two wheels of a carriage are respectively 14.25 ft. and 12.72 ft. in circumference. How many times oftener does the smaller turn than the larger in going a mile?

11. Divide 5.25 by .75, and .75 by 5.25; multiply the sum of the quotients by their difference.

12. Simplify $\dfrac{(3.2 + .004 - 1.111) \times .25}{4 \div .2 - 17.907}$.

13. Show that $.03 < \frac{11}{200}$ and $> \frac{4}{170}$.

14. Convert $\dfrac{.\dot{5} \div 3.25}{.075} \times .00025$ into a single decimal.

15. Simplify $\dfrac{.0005}{2.5} + \dfrac{.06}{.0002}$.

16. What is the greatest number of bins, each holding 12.375 bu., that can be filled from 1000 bu. wheat, and how much remains?

DECIMAL FRACTIONS CHANGED TO COMMON FRACTIONS

$5 = \frac{5}{10} = \frac{1}{2}$; $.75 = \frac{75}{100} = \frac{3}{4}$; $.035 = \frac{35}{1000} = \frac{7}{200}$;

$$.3 = \frac{3}{10} = \frac{1\frac{1}{2}}{5} = \frac{\frac{3}{5}}{2}; \qquad 12.6 = 12\tfrac{6}{10} = 12\tfrac{3}{5}.$$

It is easy to see that all decimal fractions may be changed to common or vulgar fractions without any change of value.

State the process by which such changes are made.

COMMON FRACTIONS CHANGED TO DECIMAL FRACTIONS

All common fractions may be changed to decimal fractions without change of value.

$\frac{3}{4} = .75.$ Proof: $\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = .75.$ Why?

$\frac{5}{8} = .625.$ Proof: $\frac{5}{8} = \frac{5 \times 125}{8 \times 125} = \frac{625}{1000} = .625.$ Why?

State short rule for changing a common fraction to a decimal fraction.

$\frac{1}{2} = .5$; $\frac{1}{3} = .333 \cdots$; $\frac{1}{4} = .25$; $\frac{1}{5} = .2$; $\frac{1}{6} = .166 \cdots$; $\frac{1}{7} = .142857142857 \cdots$; $\frac{1}{8} = .125$; $\frac{1}{9} = .111 \cdots$; $\frac{1}{11} = .0909 \cdots$; $\frac{1}{12} = .0833 \cdots$.

Some of the above common fractions give rise to *terminating* decimals; thus, $\frac{3}{5} = .6$, and $\frac{2}{5} = .4$. Some give rise to *repeating* decimals; thus, $\frac{1}{3} = .333 \cdots$; the "3" repeating without limit.

Repeating decimals are sometimes called *circulating* or *recurring* decimals.

To indicate $\frac{1}{3}$ exactly in the decimal form, a dot is placed over the repeating figure; thus, $\frac{1}{3} = .333 \cdots = .\dot{3}$; $\frac{5}{6} = .8333 \cdots = .8\dot{3}$; $\frac{2}{9} = .222 \cdots = .\dot{2}$; $\frac{3}{11} = .2727 \cdots = .\dot{2}\dot{7}$.

$\frac{4}{7} = .571428571428 \cdots = .\dot{5}7142\dot{8}$. Where a *group* of figures circulates, or recurs, dots are placed over the first and last figures of the group.

Reduce $\frac{2}{7}$, $\frac{3}{7}$, and $\frac{6}{7}$ to circulating decimals, and make observations as to the figures in the group.

Reduce $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, $\frac{4}{11}$, $\frac{5}{11}$, $\frac{6}{11}$, $\frac{7}{11}$, $\frac{8}{11}$, $\frac{9}{11}$, and $\frac{10}{11}$ to circulating decimals and observe the results.

A decimal fraction which contains no figures except those that recur is called a *pure* circulate; thus, $\frac{1}{3} = .\dot{3}$; $\frac{5}{9} = .\dot{5}$; $\frac{8}{11} = .\dot{7}\dot{2}$; $\frac{5}{13} = .\dot{3}8461\dot{5}$.

A decimal fraction which contains other figures than the recurring ones is called a *mixed* circulate; thus, $\frac{1}{6} = .1\dot{6}$; $\frac{5}{12} = .41\dot{6}$; $\frac{8}{15} = .5\dot{3}$; $\frac{5}{18} = .2\dot{7}$.

Decimals which do not repeat are called *finite*.

PROPERTIES OF COMMON FRACTIONS WHEN IN THEIR LOWEST TERMS

I. All common fractions whose denominators contain no factor except 2 or 5 give rise to *terminating* decimals; thus, $\frac{8}{25} = .32$, and $\frac{15}{16} = .9375$.

II. Those whose denominators do not contain 2 or 5, but factors prime to these, give rise to *pure* circulates; thus, $\frac{3}{13} = .\dot{2}3076\dot{9}$, and $\frac{9}{11} = .\dot{8}\dot{1}$.

III. Those whose denominators contain 2 or 5, *and* some factor prime to these, give rise to *mixed* circulates; thus, $\frac{5}{6} = .8\dot{3}$, $\frac{7}{12} = .58\dot{3}$, and $\frac{11}{24} = .458\dot{3}$.

The number of places in the repeating portion of a decimal fraction is always less than the number of units in the denominator of the common fraction which gives rise to it. For, in any division, the number of possible remainders is equal to the number of units in the divisor less 1. Why?

To change a pure circulate to an equivalent common fraction:

1. $.\dot{5} = \frac{5}{9}$. Proof:

$$\begin{array}{l}
.\dot{5} \times 10 = .555 \cdots \times 10 = 5.\dot{5} \\
.\dot{5} \times 1 = .\dot{5} \\
\hline
.\dot{5} \times 9 = 5 \\
\therefore\ .\dot{5} = \frac{5}{9}.
\end{array}$$

2. $.\dot{4}\dot{5} = \frac{5}{11}$. Proof:

$$\begin{array}{l}
.\dot{4}\dot{5} \times 100 = .4545 \cdots \times 100 = 45.\dot{4}\dot{5} \\
.\dot{4}\dot{5} \times 1 = .\dot{4}\dot{5} \\
\hline
.\dot{4}\dot{5} \times 99 = 45 \\
\therefore\ .\dot{4}\dot{5} = \frac{45}{99} = \frac{5}{11}.
\end{array}$$

Hence, to change a *pure* circulate to an equivalent common fraction, write the figures of the circulate for the numerator, and for the denominator write as many 9's as there are places in the circulate. Thus, $.\dot{2} = \frac{2}{9}$, and $.\dot{3}2\dot{4} = \frac{324}{999}$.

To change a mixed circulate to an equivalent common fraction:

1. $.58\dot{3} = \frac{7}{12}$. Proof:

$$\begin{array}{l}
.58\dot{3} \times 1000 = 583.\dot{3} \\
.58\dot{3} \times 100 = 58.\dot{3} \\
\hline
.58\dot{3} \times 900 = 525. \\
\therefore\ .58\dot{3} = \frac{525}{900} = \frac{7}{12}.
\end{array}$$

2. $.1\dot{6} = \frac{1}{6}$. Proof:

$$\begin{array}{l}
.1\dot{6} \times 100 = 16.\dot{6} \\
.1\dot{6} \times 10 = 1.\dot{6} \\
\hline
.1\dot{6} \times 90 = 15. \\
\therefore\ .1\dot{6} = \frac{15}{90} = \frac{1}{6}.
\end{array}$$

Hence, to change a *mixed* circulate to an equivalent common fraction, subtract the finite portion from the ole decimal, and write the result for a numerator;

and for the denominator, write as many 9's as there are figures in the circulate, and annex as many 0's as there are finite places. Thus,

$$.64\dot{7}\dot{2} = \frac{6472 - 64}{9900} = \tfrac{6408}{9900}; \ .100\dot{4} = \frac{1004 - 100}{9000} = \tfrac{904}{9000}.$$

EXERCISES

1. Show $.\dot{0}\dot{7} = \frac{7}{99}$; $.1\dot{5}\dot{6} = \frac{155}{990}$; $.0\dot{7}5\dot{6} = \frac{756}{9990}$.

2. Reduce to equivalent common fractions in their lowest terms: $.5\dot{7}$; $.04\dot{8}$; $.65\dot{9}\dot{0}$; $5.2\dot{7}$.

3. Reduce in like manner: $.\dot{6}\dot{6}$; $.\dot{9}512\dot{1}$; $\dot{1}5.\dot{0}$; $.\dot{2}7\dot{9}$; and $7.01\dot{2}\dot{6}$.

OPERATIONS UPON CIRCULATING DECIMALS

The same operations may be performed upon circulates as upon simple numbers or upon common fractions; that is, the operations of addition, subtraction, multiplication, and division.

The simplest way of performing these operations is to change the circulates to equivalent common fractions; perform the indicated operation; then change the result to the decimal form, using the circulate notation, if necessary.

EXERCISES

1. $.5\dot{3} + .\dot{4} = \frac{48}{90} + \frac{4}{9} = \frac{48}{90} + \frac{40}{90} = \frac{88}{90} = .9\dot{7}$. $\therefore .5\dot{3} + .\dot{4} = .9\dot{7}$.
2. Add $.\dot{5}$, $.\dot{3}\dot{2}$, and $.12$, giving answer as a circulate.
3. Add $2.\dot{4}$, $.\dot{5}6\dot{7}$, and $.2\dot{1}$, giving answer as a circulate.
4. Add $.\dot{4}\dot{5}$, $8.\dot{3}1\dot{8}$, $.86\dot{3}$, and $.\dot{0}1\dot{6}$.
5. From $.\dot{3}8\dot{2}$ take $.\dot{0}\dot{7}$, expressing the result as a circulate.
6. From $.\dot{2}9\dot{7}$ take $.1\dot{4}\dot{5}$, expressing the result as a circulate.
7. Multiply $.60\dot{4}$ by $.2\dot{3}\dot{4}$ and divide result by $.00\dot{4}$.

Solution. Multiply $.60\dot{4}$ by $.2\dot{3}\dot{4}$ and divide result by $.00\dot{4} = \frac{544}{900} \times \frac{232}{990} \div \frac{4}{900} = \frac{544}{900} \times \frac{232}{990} \times \frac{900}{4} = \frac{68 \times 232}{495} = 31.8\dot{7}\dot{0}$.

ACCOUNTS AND BILLS

An *account* is a record of debts and credits between persons having business relations with each other.

A *debt* is that which one person owes to another.

A *credit* is a sum due one person from another.

A *debtor* is a person owing a debt.

A *creditor* is a person to whom the debt is owed.

The *balance* of an account is the difference between the debts and credits.

A *bill* is a written statement given to the buyer by the seller, containing the number and kind of each article sold, as well as the price.

A bill is said to be *receipted* when the words "Received Payment" are written at the close of the bill, and this followed by the name of the creditor, or some person authorized by him.

The following abbreviations are in common use:

@	at.	Cr.	creditor.
a/c	account.	Do.	the same.
Acct.	account.	Dr.	debtor.
Bal.	balance.	Mdse.	merchandise.
Bo't	bought.	Pay't	payment.
Co.	company.	Pd.	paid.
¢	cents.	Rec'd	received.

Form of a Bill

Buffalo, June 3, 1902.

H. C. Spooner,

Bo't of D. R. Coate.

54 lbs. Butter	@	*25¢*	*$ 13.50*	
120 lbs. Pecans	@	*12¢*	*14.40*	
245 boxes Lobsters	@	*42¢*	*102.90*	
96 lbs. C. soap	@	*7½¢*	*7.20*	
124 lbs. Shrimp	@	*37½¢*	*46.50*	
42½ lbs. Persian Dates	@	*8¢*	*3.40*	
80 lbs. Elgin B. Butter	@	*24¢*	*19.20*	
129 doz. Pineapples	@	*$ 1.12*	*144.48*	
192 lbs. N.Y.C. cheese	@	*17½¢*	*33.60*	
84 qt. R. Whiskey	@	*$ 1.08*	*90.72*	*$ 475.90*

Rec'd payment,

D. R. Coate.

Cincinnati, Ohio, June 3, 1902.

W. R. Ball,

Bo't of Simeon Wagard.

9 yd. Silk	@	*$ 0.95*		
12 yd. Muslin	@	*0.15*		
6 yd. Doeskin	@	*1.12½*		
2 Cravats	@	*1.25*		
½ doz. Sleeve Buttons	@	*0.48*		
3 Vests	@	*2.40*		
2⅓ doz. Collars	@	*2.25*		*$ 32.29*

EXERCISES

1. On May 30, 1902, A. H. Hall bought 18 yd. drilling at $12\frac{1}{2}$¢; 30 yd. gingham at 25¢; $2\frac{1}{2}$ yd. silk velvet at $4.00; 2 silk handkerchiefs at $1.10; 16 yd. jeans at 75¢; 15 doz. wool hose at $3.00; 22 yd. red flannel at $62\frac{1}{2}$¢. Make out a receipted bill.

2. On June 4, 1902, H. L. Mewes bought of Samson Leeds, 60 oz. morphine at $3.12; 18 lb. insect powder at 32¢; 3 lb. blue mass at 96¢; 28 lb. potassium bromide at 64¢; 12 lb. senna leaves at 49¢; 70 lb. nitric acid at 14¢; 36 lb. gum camphor at 54¢. Make out a bill.

3. On May 25, 1902, M. C. Katt bought of R. H. Gaston, 7435 ft. hemlock at $12.50 per M; 8416 ft. pine flooring at $23.20 per M; 4568 ft. clear pine at $44.50 per M; 2976 ft. oak joists at $3.25 per C; 7814 ft. ash flooring at $3.40 per C. Make out a bill in proper form and receipt it.

INVOLUTION

The process of raising a number to a given power is called *involution*.

A power of a number is the number itself, or the result obtained by using the number one or more times as a factor. Thus, 2, 4, 8, 16, 32, are the first, second, third, fourth, and fifth powers of 2. 3, 9, 27, 81, are the first, second, third, and fourth powers of 3.

A power of a number is indicated by an exponent, which is a small figure or letter placed to the right and a little above the number; thus 5^2 is read, "the second power of 5"; and a^m is read, "a to the mth," or "the mth power of a."

There is very little difficulty in finding a given power of a number, especially if the number is expressed in Arabic notation, and the exponent is a positive integer.

EXERCISES

1. What is the square of 29? Of 3.01? Of 1.01? Of .003?

2. What is the third power of 1.8? Of .203? Of .1009?

3. $.1^3 = ?$ $\overline{.089}^2 = ?$ $\overline{17.5}^2 = ?$ $\overline{.02}^6 = ?$ $\overline{.005}^4 = ?$ $(\frac{1}{6})^5 = ?$

4. What is the fourth power of 3? Of .02? Of $2\frac{1}{2}$? Of $\frac{2}{5}$?

NOTE. The squares of all numbers from 1 to 25 and the cubes of all numbers from 1 to 10 should be committed to memory.

EVOLUTION

Evolution is the process of finding a root of a number, and is the *inverse* of involution. A root of a number is one of the equal factors of it. The square root is one of the *two* equal factors of it; the cube root one of the *three* equal factors of it.

Thus, 5 is one of *two* equal factors of 25, and 6 is one of the *three* equal factors of 216; for $5 \times 5 = 25$, and $6 \times 6 \times 6 = 216$. $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$; hence the cube root of $\frac{1}{27} = \frac{1}{3}$.

The symbol $\sqrt{\ }$ is called a radical sign and aids in indicating a root of a number. Thus, $\sqrt{49} = 7$; $\sqrt{\frac{4}{9}} = \frac{2}{3}$; $\sqrt[3]{64} = 4$; $\sqrt[6]{64} = 2$; $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$. These are read, the square root of $49 = 7$, the square root of $\frac{4}{9} = \frac{2}{3}$, the cube root of $64 = 4$, the sixth root of $64 = 2$, and the fifth root of $\frac{1}{32} = \frac{1}{2}$.

There is so little use of evolution in practical business that no more evolution should be studied in the public schools below the high school than can be handled mentally, or by inspection.

Thus, $\sqrt{729} = \sqrt{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = 3 \cdot 3 \cdot 3 = 27$, by definition of square root.

$\sqrt[6]{15625} = \sqrt[6]{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5} = 5$, by definition of sixth root.

$\sqrt[3]{15625} = \sqrt[3]{25 \times 25 \times 25} = 25$, by definition of cube root.

$\sqrt{15625} = \sqrt{125 \times 125} = 125$, by definition of square root.

$\sqrt[4]{(3)^8} = \sqrt[4]{(3^2)^4} = 3^2 = 9.$

$\sqrt[5]{a^{10}m^5} = \sqrt[5]{(a^2)^5 m^5} = \sqrt[5]{(a^2m)^5} = a^2m$, by definition.

$\sqrt[2]{448} = \sqrt[2]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7} = 2 \cdot 2 \cdot 2\sqrt{7} = 8\sqrt{7}.$

Some numbers cannot be factored *readily*, and some are not perfect powers of any rational number. Methods of finding the square and the cube root of such numbers will be given, but the proofs of these methods will not be given here, since they are much easier by algebraic methods.

SQUARE ROOT

$3^2 = 9$; $9^2 = 81$; $\overline{15}^2 = 225$; $\overline{99}^2 = 9801$; $\overline{110}^2 = 12{,}100$; $\overline{900}^2 = 810{,}000$. By observing these problems, and by further trial, it is found that the square of a number never contains more than twice as many figures as the number itself. Hence, point off the given number into periods of *two* figures each, beginning at units' place, and counting to the left. If the number is a decimal, or contains a decimal, point off the decimal into periods of two figures each, beginning at units' place and counting to the right.

Find the greatest square in the first period at the left; take the square root of it for the first figure in the root; subtract this square from the left-hand period; then bring down the remainder and the next period for a new dividend.

Double the root already found for a trial divisor; find how many times the trial divisor is contained in the dividend, omitting the right-hand figure; write the quotient

as the second figure of the root, and also annex it to the *trial* divisor for a *complete* divisor. Multiply this complete divisor by the last number of the root. Subtract the result from the dividend, and to the remainder annex the next period, and proceed as before.

EXERCISES

1. Find the square root of 10,795.21.

SOLUTION

	1,07,95.21	(103.9
$1 \times 1 =$	1	
$20 \times 0 = 0$	795	
$203 \times 3 =$	609	
	186.21	
$206.9 \times .9 =$	186.21	
	000.00	

$$\therefore \sqrt{10796.21} = 103.9.$$

2. Find the square root of 15,239.9025.

SOLUTION

	1,52,39.90,25	(123.45
$1 \times 1 =$	1	
	52	
$22 \times 2 =$	44	
	839	
$243 \times 3 =$	729	
	110.90	
$246.4 \times 4 =$	98.56	
	12.3425	
$246.85 \times 5 =$	12.3425	
	00.0000	

$$\therefore \sqrt{15239.9025} = 123.45.$$

CUBE ROOT

$2^3 = 8$; $9^3 = 729$; $\overline{11}^3 = 1331$; $\overline{90}^3 = 729{,}000$; $\overline{110}^3 = 1{,}331{,}000$; $\overline{900}^3 = 729{,}000{,}000$.

Observe that the cube of a number cannot have more than three times as many figures as the figure itself, nor but two less.

Hence, if a number is separated into periods of three figures each, beginning at units' place, the number of figures in the cube root will be the same as the number of periods.

Hence, to find the cube root of a number point it off into periods of *three* figures each, counting from units' place each way.

Find the cube root of the largest perfect cube in the left-hand period for the first figure of the root; subtract this perfect cube from the left-hand period, and bring down the remainder in connection with the next period for a new dividend.

Square the number represented by the first figure in the root, and multiply this result by 300 for a trial divisor; divide the new dividend by this trial divisor for the second figure of the root. To the trial divisor add the continued product of the first figure of the root by the second and by 30.

To this sum add the square of the number represented by the second figure of the root, which gives the complete divisor.

Multiply the complete divisor by the second figure of the root. Subtract the product from the dividend and bring down the remainder in connection with the next period for another dividend, and proceed as before, treating the portion of the root already found as "the first figure of the root."

EXERCISES

1. Find the cube root of 15,625.

Solution

$$
\begin{array}{rrl}
 & & 15{,}625(\underline{25} \\
 & 2 \times 2 \times 2 = & \underline{8} \\
 & & 7625 \\
2 \times 2 \times 300 = 1200 & & \\
2 \times 5 \times 30 = 300 & & \\
5 \times 5 = \underline{25} & & \\
 & 1525 \times 5 = & \underline{7625} \\
 & & 0000 \quad \therefore \sqrt[3]{15{,}625} = 25.
\end{array}
$$

2. Find the cube root of 134,217,728.

Solution

$$
\begin{array}{rrl}
 & & 134{,}217{,}728(\underline{512} \\
 & 5 \times 5 \times 5 = & \underline{125} \\
 & & 9{,}217 \\
5 \times 5 \times 300 = 7500 & & \\
5 \times 1 \times 30 = 150 & & \\
1 \times 1 = \underline{1} & & \\
 & 7651 \times 1 = & \underline{7{,}651} \\
 & & 1{,}566{,}728 \\
51 \times 51 \times 300 = 780{,}300 & & \\
51 \times 2 \times 30 = 3{,}060 & & \\
2 \times 2 = \underline{4} & & \\
 & 783{,}364 \times 2 = & \underline{1{,}566{,}728} \\
 & & 0\,000\,000 \\
 & & \therefore \sqrt[3]{134{,}217{,}728} = 512.
\end{array}
$$

EXERCISES

Find the square root of the following:

1.	2916.	**4.**	.0784.	**7.**	.00953361.
2.	531,451.	**5.**	.766961.	**8.**	3685.000289.
3.	287.65.	**6.**	1073.741.	**9.**	$\frac{578}{1250}$.

Find the cube root of the following:

10.	39,304.	**12.**	109,095.488.	**14.**	125,000,512.	**16.**	$\frac{729}{2744}$.
11.	1.74088.	**13.**	46,268,279.	**15.**	$\frac{125}{216}$.	**17.**	$\frac{3456}{8192}$.

GEOMETRIC ILLUSTRATION OF SQUARE ROOT

Find the square root of 2116.

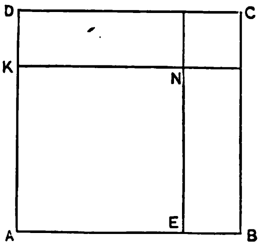


Fig. 1.

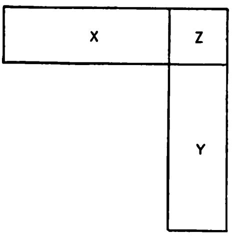


Fig. 2.

Suppose *ABCD* (Fig. 1) is a square, and that it contains 2116 square units.

It is required to find the length of one side, as *AB*, in linear units.

A square whose side is 40 units long contains 1600 square units, and one whose side is 50 units long contains 2500 square units. Hence, the side of this square must contain between 40 and 50 linear units.

Suppose a square whose side is 40 units long, as *AENK*, is taken out. Then 1600 square units are taken out and

516 square units remain in the irregular figure X, Y, Z (Fig. 2).

The rectangles X and Y and the square Z may be arranged as in Fig. 3, whose width is wanted. The rectangles X and Y are each 40 units long; hence, the rectangle XYZ is over 80 units long. The width cannot be 7 units, for then the area would be over 560 square units. Try 6 units for the width; then the length is 86 units, for Z is a square. The area of a rectangle 86 units long and 6 units wide is 516 square units. But this is the known area of the remainder of the square $ABCD$.

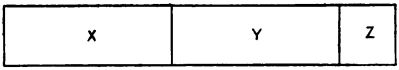


FIG. 3.

$\therefore$ the length of AB is $(40 + 6) = 46$ units.

Check: $\overline{46}^2 = 2116$.

GEOMETRIC ILLUSTRATION OF CUBE ROOT

Find the cube root of 32,768.

Suppose the cube $ABCD$ contains 32,768 cubic units. To find the number of linear units in one edge.

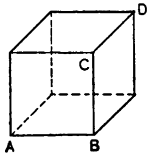


FIG. 1.

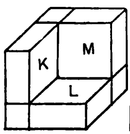


FIG. 2.

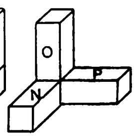


FIG. 3.

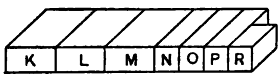


FIG. 4.

A cube whose edge contains 30 linear units contains 27,000 cubic units, and one whose edge is 40 units long contains 64,000 cubic units (Fig. 1). Hence, one edge is over 30 and less than 40 units long. Take away a cube whose edge is 30 units long, containing 27,000 cubic units, and the portion remaining contains 5768 cubic units, as Fig. 2.

The three solids *K*, *L*, and *M* are each 30 units square, and the thickness is unknown.

Take away these three, and there remains a portion which is shown in Fig. 3.

The three portions *N*, *O*, and *P* are each 30 units long, and as wide and thick as the thickness in solids *K*, *L*, and *M*.

Remove *N*, *O*, and *P*, and a cube *R* remains, shown in Fig. 4, each of whose edges contains as many units as the thickness of *K*, *L*, and *M*.

Now the portion of the original cube which contains 5768 cubic units may be made to take the form of Fig. 4.

$\because$ the area of the base of each *K*, *L*, and *M* is 900 square units, the three bases contain 2700 square units. If the altitude of the solid in Fig. 4 is 3 units, the solids *K*, *L*, and *M* would contain 8100 cubic units, which is too much. Try 2 units.

Then the base of each *N*, *O*, and *P* is 30 by 2 units, or 60 square units, the bases of the three containing 180 square units. The base of *R* is 4 square units. Then the base of the solid in Fig. 4 contains 2884 square units, and the solid contains 5768 cubic units.

$\therefore$ *AB*, the edge of the cube, is $30+2$, or 32 units long.

Check: $\overline{32}^3 = 32{,}768$.

EXERCISES

1. A tower is 240 ft. high and stands 160 ft. from a flagpole which is 120 ft. high; how many feet along a straight line from the top of the tower to the top of the flagpole?

2. A pendulum which is 65 inches long swings from a point *A* to a point *B*, and *A* and *B* are in a horizontal line. If the line *AB* is 6 inches above the lowest point in the sweep, find the distance from *A* to *B*.

3. A rectangular room is 20 ft. long and 15 ft. wide. Find the length of the diagonal of the floor.

4. If the height of the room in Question 3 is 9 ft., find the length of the line from one lower corner to the opposite upper corner.

5. Rain falling on a roof which slants 45° with the horizon is collected into a vessel with a rectangular base 5 ft. long and $3\frac{1}{5}$ ft. wide, and fills the vessel to a depth of $3\frac{1}{8}$ ft.; if the rainfall was $\frac{3}{4}$ of an inch, find the area of the roof.

6. If 29,791 cubical blocks 1 cm. long are piled up in the form of a cube, how long is one edge of the pile in centimeters?

7. If a cubic foot of water weighs 1000 oz. find the length of a cubical vessel which holds $1\frac{4}{5}$ tons of water.

8. Find the length of one edge of a cubical block which contains 96 board feet.

9. Extract the cube root of $.58\dot{3}$ to 3 places.

10. Extract the square root of $\frac{20}{405}$, and of $\dfrac{22\frac{1}{2}}{40}$.

11. Extract the cube root of $\frac{128}{250}$, and of $\dfrac{90}{213\frac{1}{8}}$.

12. Extract the square root of $\frac{2}{13}$ to the nearest 1000th.

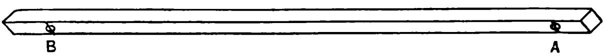
13. Extract the cube root of $\frac{49}{210}$ to the nearest 100th.

14. Extract the cube root of $21\frac{7}{8}$ to the nearest 10th.

THE ENGLISH SYSTEM OF WEIGHTS AND MEASURES

The primary unit of this system is the yard.

In the Standards Office at London there is a bronze bar 38 inches long and 1 inch square.

Thus, 

At *A* and *B* gold studs are inserted. They are circular in form. Two parallel lines across these studs are drawn. The distance between these lines is defined as the *yard*.

The length of the yard as thus defined was found by the "pendulum experiment." A pendulum was found that would beat seconds at a temperature of 62° Fahr., in a vacuum, at the level of the sea, and in the latitude of London. This experiment was authorized by the English government. The $\frac{360000}{391393}$ of the length of this pendulum was adopted as the yard, and is recorded on the bronze bar as referred to above.

If there had been no previous record, the length of the whole pendulum would, without doubt, have been made the primary unit of the English System of Weights and Measures.

A pendulum which beats seconds under the conditions mentioned above must be of constant length. Therefore $\frac{360000}{391393}$ of it must be of constant length. This unit is dependent upon the element of time; hence, it is a natural

unit, and if the record of it should be lost, another unit of the same length could be obtained.

The yard, as a unit of measure, has a very interesting history.

At one time the length which would measure around the waist of a man was taken as the yard. This was called the *girth*.

In the reign of King Henry I, he decreed that the yard should be the length of his arm measured from the center of the breast to the tip of the finger.

In the reign of Edward II, he decreed that 3 barleycorns, rounded and dried, and placed end to end, should make an inch, and that 36 inches should make a yard.

Three times the length of the human foot was called the yard.

The width of the palm of the hand taken 9 times was at one time taken as the yard.

The span, or the distance designated by the thumb and little finger stretched out, was taken 4 times to make the yard.

These and many other devices were resorted to in order to get some standard of length, but none was reliable. The pendulum experiment was made because no exact and reliable standard unit of length existed.

UNITS OF LENGTH

The yard is the primary unit of length. Other units of length are: foot, inch, line, hairbreadth, rod, furlong, mile, league, palm, hand, span, finger-length, knot, fathom, cable-length, cubit, link, chain, girth, perch, pole, barleycorn, nail, quarter, pace, military pace, Roman pace, English ell, French ell, Flemish ell, Scotch ell, leap, size, etc.

These units of length were originally derived from

natural objects of the material world. In most instances the name is suggestive of the origin of the unit.

The foot is $\frac{1}{3}$ of the yard and was originally the length of the human foot. The inch is $\frac{1}{12}$ of the foot and was originally the length of the terminal joint of the thumb.

The line is $\frac{1}{12}$ of an inch, and the hairbreadth is $\frac{1}{48}$ of an inch. The rod is $5\frac{1}{2}$ yd. long and was originally the length of "a measuring stick."

The furlong (obsolete) is 40 rods and was a furrow-length. The mile is 8 furlongs or 320 rods, and is from *mille*, a thousand, the distance of a thousand paces. The palm is 4 inches, and was the length of the palm. The hand is 4 inches and was the width of the hand. The span is 9 inches, and was the reach of the thumb and the little finger.

The finger-length is 4 inches, and was the length from the knuckle to the tip of the finger.

The knot, or geographical mile, is 1.15 statute miles.

The fathom is 6 feet, and was the length of two arms extended.

The cable-length is 120 fathoms, or 720 feet in length.

The cubit is of various lengths; the common cubit 18 inches, and the sacred cubit 22 inches, being the length of the forearm. The cubit is the most ancient unit named in sacred history.

The link is 7.92 inches, and 100 links make a chain.

The chain is rarely used now; a steel tape is used in its place.

80 chains make one mile.

The perch and the pole are the same as the rod.

The nail is $2\frac{1}{4}$ inches, the quarter 4 nails, and the yard is 4 quarters. The nail and the quarter are obsolete units now, but were once used in cloth measure.

The English ell, the Scotch ell, the Flemish ell, and the French ell are now practically obsolete, but were once units of cloth measure, and they varied in length from 27 in. to 54 in.

The common pace is 3 ft., the military pace is $2\frac{1}{2}$ ft., and the Roman pace is 5 ft., the latter being the distance passed over by one foot from one position to the next.

The size as used by shoemakers is historically $\frac{1}{3}$ of an inch.

Some of the units mentioned above are obsolete and many others are miscellaneous in their nature. The units most used in practice are tabulated below:

12	inches (in.)	= 1 foot (ft.)
3	ft.	= 1 yard (yd.)
$5\frac{1}{2}$	yd.	= 1 rod (rd.)
320	rd.	= 1 mile (mi.)

The scale is not uniform, being 12, 3, $5\frac{1}{2}$, and 320.

UNITS OF SURFACE MEASURE

A surface unit is, in general, a portion of surface in the form of a square, each side of which is some unit of length.

Hence, the surface unit is derived from and determined by the linear unit.

The history of these units is of no importance. The most prominent ones are here tabulated.

144	square inches (sq. in.)	= 1 square foot (sq. ft.)
9	sq. ft.	= 1 square yard (sq. yd.)
$30\frac{1}{4}$	sq. yd.	= 1 square rod (sq. rd.)
160	sq. rd.	= 1 acre (A.)
640	A.	= 1 square mile (sq. mi.)
36	sq. mi.	= 1 township (Tp.)

The rood is equal to 40 sq. rd., and 4 roods equal 1 A. But the rood is almost obsolete as a unit. 10,000 square links = 1 square chain, and 10 sq. ch. = 1 A.

UNITS OF VOLUME MEASURE

A unit of volume is, in general, a portion of a solid in the form of a cube, each edge of which is some unit of length. Hence, it is derived from and determined by the linear unit, and has no history of importance.

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cu. ft.	= 1 cubic yard (cu. yd.)
128 cu. ft.	= 1 cord (8 ft. by 4 ft. by 4 ft.)
16 cu. ft.	= 1 cord foot

40 cu. ft. of hewn timber, or 50 cu. ft. of round timber, = 1 ton or load. $24\frac{3}{4}$ cu. ft. = 1 perch of stone. 1 cu. yd. of earth is usually called a wagon load.

UNITS OF CAPACITY

Dry Measure. The primary unit is the bushel, which is equal to 2150.42 cu. in.; hence, this unit is derived from and is determined by the yard. The word "bushel" is derived from a word meaning box; hence, a bushel was a boxful.

The derived units of Dry Measure are the peck, the gallon, the quart, and the pint.

The word "peck" is a corruption of the word "pack."

The word "quart" is from a word meaning fourth part. The word "gallon" is from *galon*, a grocer's box. The word "pint" is from *pinto*, a drink.

2 pints (pt.)	= 1 quart (qt.)
4 qt.	= 1 gallon (gal.)
2 gal.	= 1 peck (pk.)
4 pk.	= 1 bushel (bu.)

The stricken bushel is used in measuring shelled grain. The heaped bushel is used in measuring corn in the ear, coal, fruit, etc. The heaped bushel $=1\frac{1}{6}$ stricken bushels.

Liquid Measure. The primary unit is the wine gallon, which contains 231 cubic inches. Hence, it is derived from and is dependent upon the yard. The derived units of Liquid Measure are the gill, the pint, the quart, the barrel, and the hogshead. These units were primarily for wine measure, but now they are used for measuring all kinds of liquids.

The word "gill" is from *gilla*, a drinking-glass.

4 gills	= 1 pint (pt.)
2 pt.	= 1 quart (qt.)
4 qt.	= 1 gallon (gal.)
$31\frac{1}{2}$ gal.	= 1 barrel (bbl.)
2 bbl.	= 1 hogshead (hhd.)

In ale, beer, or milk measure (now obsolete), 282 cu. in. = 1 gal., 36 gal. = 1 bbl., $1\frac{1}{2}$ bbl. = 1 hhd.

UNITS OF WEIGHT

Troy Weight. The primary unit is the pound (lb.). The word "pound" is from the word *pendo*, I bend,—an object weighing more or less according to the extent it would bend a stick. At present the troy pound is defined by the weight of 22.794377 cu. in. of distilled water at a temperature of about 40° Fahr.

The derived units are the ounce, the pennyweight, and the grain.

The word "ounce" is from the word *uncia*, the twelfth part; the pennyweight was the weight of an English penny; the grain was the weight of a grain of wheat taken from the middle of the head.

24 grains (gr.)	= 1 pennyweight (pwt. or dwt.)	
20 pwt.	= 1 ounce (oz.)	
12 oz.	= 1 pound (lb.) = 5760 grains	

The troy grain was originally the base of all English weights. The word "troy" is supposed to be derived from *Troyes*, a town in France, where it was first used in Europe. By some it was thought to mean simply London weight.

These troy units are used in weighing the more precious materials, as gold, silver, platinum, jewels, etc.

Avoirdupois Weight. The primary unit is the pound, which is the weight of 27.7274 cu. in. distilled water at a temperature of 62° Fahr. The troy and avoirdupois pounds are both derived from and dependent upon the yard. The avoirdupois pound contains 7000 troy grains.

Hence, the avoirdupois pound contains 1240 grains more than the troy pound. The common saying, "A pound of lead is heavier than a pound of gold," is literally true. But it is as true that an ounce of gold is heavier than an ounce of lead. Why?

A pound of feathers is heavier than a pound of platinum. Why? An ounce of platinum is heavier than an ounce of feathers. Why? Since Avoirdupois Weight is used in weighing the heavier and coarser materials, there is little use for any small units, say below the ounce. However, the drachm is sometimes used.

16 drachms	= 1 ounce (oz.)
16 oz.	= 1 pound (lb.)
100 lb.	= 1 hundredweight (cwt.)
20 cwt.	= 1 ton (T.)

The grain is the common unit of comparison between troy and avoirdupois units.

Diamond Weight. The primary unit is the carat, which equals 4 grains. But the grain in Diamond Weight is only $\frac{4}{5}$ of a troy grain. 1 grain equals 16 parts. 1 part = $\frac{1}{20}$ grain troy. Hence,

16 parts = 1 grain
4 grains = 1 carat

Apothecaries Weight. The primary unit is the pound troy. The derived units are the ounce, dram, scruple, and grain.

20 grains	= 1 scruple (℈),	meaning little stone
3 ℈	= 1 dram (ʒ),	meaning a piece of money
8 ʒ	= 1 ounce (℥),	meaning $\frac{1}{12}$ part
12 ℥	= 1 pound (℔),	= 5760 grains

UNITED STATES SYSTEM OF WEIGHTS AND MEASURES

In 1834 the United States government adopted a uniform standard of weights and measures for all purposes connected with the general government. Most, if not all, states have adopted the same standard.

This standard is the yard, which, until 1893, was the same as the imperial yard of Great Britain, which was obtained by the "pendulum experiment." Since 1893 the U. S. yard has been defined as equal to $\frac{3600}{3937}$ of the meter. A record of the length of the yard is kept on a metal rod at Washington, D.C., from which duplicates are furnished to the states.

The gallon, the bushel, the troy pound, the avoirdupois pound, and the carat are *primary* units; and the units derived from these are the same in the United States as in Great Britain.

In 1836 Congress approved of a uniform set of weights and measures for all of the states, which set consisted of a yard, a set of troy weights, a set of avoirdupois weights, a wine gallon and its subdivisions, and a half-bushel and its subdivisions. These standards were sent to the states in 1840. Each state sends standard sets to its counties.

For the study of weights and measures, each school should have:

1. A good balance, comprising a beam and scale dishes.

2. A bed of troy weights made of brass, namely, $\frac{1}{16}$ oz., $\frac{1}{8}$ oz., $\frac{1}{4}$ oz., $\frac{1}{2}$ oz., 1 oz., 2 oz., 4 oz., on to 50 oz. inclusive.

3. A bed of avoirdupois weights made of brass, namely, $\frac{1}{16}$ oz., $\frac{1}{8}$ oz., $\frac{1}{4}$ oz., $\frac{1}{2}$ oz., 1 oz., on to 10 lb. inclusive.

4. A brass yard measure, showing feet, inches, and divisions.

5. A steel tape 50 ft. long, showing on one side the foot and 10ths and 100ths of a foot, and on the other side the link, the rod, and the pole.

6. A nest of dry measures made of brass, comprising 1 qt., 1 half-gal., 1 gal., 1 pk., and 1 half-bu.

7. A nest of liquid measures made of brass, comprising 1 gill, 2 gills, 1 pt., 1 qt., 2 qt., and 1 gal.

A neat case should be provided for taking good care of the measures.

REDUCTION OF DENOMINATE NUMBERS

Reduction is the operation of changing an expression in one or more denominations to an equivalent expression in some other denomination or denominations in the same system of measure. Reduction does not mean *to make smaller* in the technical use of the word. It means a change or transformation from a given denomination to a *higher* or to a *lower* denomination.

Thus, 8 ft. may be reduced to inches, reduction to a lower denomination; and 250 gr. may be reduced to ounces, reduction to a higher denomination.

A simple denominate number may be reduced to a lower simple denominate number; thus, 2 rd. = 33 ft.

A compound denominate number may be reduced to a simple denominate number; thus, 2 oz. 3 pwt. 5 gr. = 1037 gr.

A fractional denominate number may be reduced to a simple number of lower denomination; thus, $\frac{5}{8}$ gal. = 5 pt., and .125 gal. = 4 gills.

A fractional denominate number may be reduced to a compound number of lower denominations; thus, $\frac{2}{7}$ lb. troy = 3 oz. 8 pwt. $13\frac{5}{7}$ gr., and .325 bu. = 1 pk. 2 qt. .8 pt.

A simple denominate number may be reduced to a com-

pound number of higher denominations; thus, 271 pt. = 33 gal. 3 qt. 1 pt.

A simple denominate number may be reduced to a fractional number of higher denomination; thus, 5 gills = $\frac{5}{32}$ gal., and 5 pwt. = .25 oz.

A compound denominate number may be reduced to a fractional number of higher denomination; thus, 3 pwt. 5 gr. = $\frac{77}{480}$ oz., and 4 pwt. 15 gr. = .23125 oz.

A fractional number of a given denomination may be reduced to a fractional number of a higher denomination; thus, $\frac{3}{4}$ pt. = $\frac{3}{32}$ gal. = .09375 gal.

EXERCISES

1. Reduce 8 ft. to inches.

SOLUTION. 1 ft = 12 in. $\therefore$ 8 ft. = 12 in. $\times$ 8 = 96 in.

2. Reduce 8 A. to square rods.
3. Reduce 5 cords to cubic feet.
4. Reduce 85 lb. troy to grains.
5. Reduce 42 gal. wine to gills.
6. Reduce 37 bu. to quarts.
7. Reduce 24 mi. to inches.
8. Reduce 15° to seconds.
9. Reduce 145 da. to minutes.
10. Reduce £24 to pence.
11. Reduce 750 sq. ch. to acres.
12. Reduce 5 carats to parts.
13. Reduce 15 ʒ to grains.
14. Reduce 7 cu. yd. to cubic inches.

EXERCISES

1. Reduce 8 bu. 3 pk. 2 qt. to quarts.

SOLUTION

$$\begin{array}{r} 8 \text{ bu.} \quad 3 \text{ pk.} \quad 2 \text{ qt.} \\ \underline{4} \\ 32 \\ \underline{3} \\ 35 \\ \underline{8} \\ 280 \\ \underline{2} \\ 282 \end{array}$$

$\therefore$ 282 quarts.

REMARK. Before one can know what operation to perform, he must know the table for *Dry Measure*.

$\because$ 4 pk. = 1 bu., 8 bu. 3 pk = 35 pk. $\because$ 8 qt. = 1 pk., 35 pk. 2 qt. = 282 qt.

2. Reduce 9 yd. 2 ft. 5 in. to inches.

3. Reduce £128 11*s.* 7*d.* 1 far. to farthings.

4. Reduce 24 lb. 3 oz. 2 pwt. 15 gr. to grains.

5. Reduce 5 mi. 75 rd. 4 yd. 2 ft. 5 in. to inches.

6. Reduce 37 A. 25 sq. rd. 2 sq. yd. 5 sq. ft. to square feet.

7. Reduce 10 mi. 44 ch. 75 l. 4 in. to inches.

8. Reduce 75 cu. yd. 5 cu. ft. 1001 cu. in. to cubic inches.

9. How many minutes from midnight, Dec. 31, 1895, to 3 P.M., Apr. 14, 1900?

10. Find the value of 48 lb. 5 oz. 12 pwt. gold at $1.92½ per pwt.

11. Find the value of 75 A. 45 sq. rd. 15 sq. yd. at 2¢ per square yard.

12. Find the value of 3 T. 15 cwt. coal at 37½¢ per hundredweight.

13. Reduce 3 ch. 45 l. 4 in. to inches.

EXERCISES

1. Reduce £$\frac{5}{6}$ to farthings.

SOLUTION. $\frac{5}{6} \times 960 = 800$. $\therefore$ £$\frac{5}{6}$ = 800 farthings.

EXPLANATION. From the table for English Money, £1 = 20*s.* = 240*d.* = 960 far. $\therefore$ £$\frac{5}{6} = \frac{5}{6}$ of 960 far. = 800 far.

2. Reduce $\frac{3}{8}$ lb. troy to grains.
3. Reduce $\frac{2}{9}$ bu. to pints.
4. Reduce $\frac{5}{8}$ A. to square inches.
5. Reduce $\frac{4}{15}$ yd. to inches.
6. Reduce $\frac{2}{9}$ yr. to seconds.
7. Reduce $\frac{3}{16}$ gal. to gills.
8. Reduce $\frac{7}{8}$ ton av. to ounces.
9. Reduce $\frac{3}{4}$ degree to seconds.
10. Reduce $\frac{9}{10}$ mark to pfennigs.

EXERCISES

1. Reduce $\frac{3}{7}$ lb. troy to a compound denominate number.

SOLUTION.
$\frac{3}{7}$ of 12 oz. = 5 oz. and $\frac{1}{7}$ oz.
$\frac{1}{7}$ of 20 pwt. = 2 pwt. and $\frac{6}{7}$ pwt.
$\frac{6}{7}$ of 24 gr. = $20\frac{4}{7}$ gr.
$\therefore \frac{3}{7}$ lb. troy = 5 oz. 2 pwt. $20\frac{4}{7}$ gr.

2. Reduce $\frac{3}{20}$ mi. to a compound denominate number.
3. Reduce $\frac{5}{8}$ A. to a compound denominate number.
4. Reduce $\frac{3}{14}$ cd. to a compound denominate number.
5. Reduce $\frac{102}{175}$ bu. to a compound denominate number.
6. Reduce $\frac{25}{36}$ bbl. wine to a compound denominate number.
7. Reduce $\frac{250}{281}$ wk. to a compound denominate number.
8. Reduce $\frac{50}{81}$ T. av. to a compound denominate number.

EXERCISES

1. Reduce .36 bu. to lower denominations.

SOLUTION. .36 of 4 pk. = 1.44 pk. = 1 pk. and .44 pk.
.44 of 8 qt. = 3.52 qt. = 3 qt. and .52 qt.
.52 of 2 pt. = 1.04 pt.
∴ .36 bu. = 1 pk. 3 qt. 1.04 pt.

2. Reduce .375 mi. to lower denominations.
3. Reduce .84 da. to lower denominations.
4. Reduce .4975 circle to lower denominations.
5. Reduce .685 A. to lower denominations.
6. Reduce .64 lb. troy to lower denominations.
7. Reduce .245 gal. wine to lower denominations.
8. Reduce .28 ton to lower denominations.

EXERCISES

1. Reduce 450746 min. to higher denominations.

SOLUTION.
$$\begin{array}{r|l}
60 & 450746 \\ \hline
24 & 7512 + 26 \\ \hline
 & 313 \quad\; 0
\end{array}$$

In 450746 min. there are 7512 hr. and 26 min.
In 7512 hr. there are 313 da. and no hr.
∴ 450746 min. = 313 da. 26 min.

2. Reduce 8263 gills to higher denominations.
3. Reduce 8972 links to higher denominations.
4. Reduce 384,643 in. to higher denominations.
5. Reduce 267,348 sq. in. to higher denominations.
6. Reduce 14,734 pt. to higher denominations.
7. Reduce 874,635 cu. in. to higher denominations.
8. Reduce 28,349 oz. av. to higher denominations.
9. Reduce 43,462 oz. av. to higher denominations.

EXERCISES

1. Reduce 5 gills to the fraction of a gallon.

SOLUTION. 5 gills $= \frac{5}{4}$ pt. $= \frac{5}{8}$ qt. $= \frac{5}{32}$ gal.
The table for wine measures must be known.

2. Reduce 9 in. to the fraction of a rod.

3. Reduce 15 gr. to the fraction of a pound troy.

4. Reduce 5 drams to the fraction of a hundredweight av.

5. Reduce 12 sec. to the fraction of a day.

6. Reduce 7 sq. ft. to the fraction of an acre.

7. Reduce 29 cu. in. to the fraction of a cord.

8. Reduce 18″ to the fraction of a degree.

9. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of a min. to the fraction of a day.

10. Reduce 2 oz. 3 pwt. 5 gr. to the fraction of a pound troy.

11. Reduce 3 cable-lengths 5 fathoms to the fraction of a mile.

12. Reduce 12*s.* 5*d.* to the fraction of a pound.

EXERCISES

1. Reduce .275 gallon to gills.

SOLUTION. .275 of 4 qt. = 1.1 qt.; 1.1 of 2 pt. = 2.2 pt.; 2.2 pt. = 8.8 gills. ∴ .275 gal. = 8.8 gills.

2. Reduce .525 rd. to inches.

3. Reduce .42 A. to square feet.

4. Reduce .1125 lb. troy to grains.

5. Reduce .225 cd. to cubic inches.

6. Reduce .32 ch. to inches.

7. Reduce .65 ton to ounces.

8. Reduce $\frac{.18}{73}$ yr. to seconds.

EXERCISES

1. Reduce $\frac{3}{4}$ pt. to gallons.

SOLUTION. 1 pt. = $\frac{1}{2}$ qt. $\therefore$ $\frac{3}{4}$ pt. = $\frac{3}{8}$ qt.
1 qt. = $\frac{1}{4}$ gal. $\therefore$ $\frac{3}{8}$ qt. = $\frac{3}{32}$ gal. = .09375 gal.

2. Reduce .35 qt. to bushels.

SOLUTION. 1 qt. = $\frac{1}{4}$ gal. $\therefore$ .35 qt. = $\frac{7}{80}$ gal.
1 gal. = $\frac{1}{2}$ pk. $\therefore$ $\frac{7}{80}$ gal. = $\frac{7}{160}$ pk.
1 pk. = $\frac{1}{4}$ bu. $\therefore$ $\frac{7}{160}$ pk. = $\frac{7}{640}$ bu. = .0109375 bu.

3. Reduce $\frac{3}{5}$ in. to yards.

4. Reduce $\frac{7}{9}$ cu. in. to cubic yards.

5. Reduce .125 sq. in. to square yards.

6. Reduce $\frac{7}{12}$ far. to shillings.

7. Reduce .875″ to degrees.

8. Reduce .23 in. to chains.

9. Reduce $\frac{11}{12}$ gr. to ounces.

10. Reduce $\frac{7}{8}$ gill to gallons.

11. Reduce .64 oz. av. to hundredweight.

12. Reduce .745 gal. to barrels.

Horace gives the following dialogue between teacher and pupil: "Let the son of Albinus tell me if from 5 ounces ($\frac{5}{12}$ lb.) be subtracted 1 ounce ($\frac{1}{12}$ lb.), what is the remainder? Come, you can tell. 'One-third.' Good; you will be able to take care of your property."

THE METRIC SYSTEM

OR THE

FRENCH SYSTEM OF WEIGHTS AND MEASURES

A good system of weights and measures is characterized by well-defined units and a uniform scale. These the old English system did not possess. The units were not well defined, and the scale was not uniform.

In the light of scientific attainments, the English government succeeded in defining the units well, but did not make the scale uniform. No two tables have the same scale, and in no one table is the scale uniform.

The French system is the result of attempts by the French government and the Academy of Sciences to provide something better.

The first movement toward it was in 1528, but nothing of importance was done till 1790, when the Academy of Sciences appointed a committee of five, sanctioned by the French government, to report upon the selection of a *natural standard.* The members of this committee were: Lagrange, Laplace, Borda, Monge, and Condorcet — five of the most noted mathematicians of all Europe.

Two standards were considered:

(1) The length of a pendulum which should vibrate seconds at a given point on the surface of the earth.

(2) A well-defined portion of the arc of a certain meridian.

In its report, this committee proposed that the ten-millionth part of a quadrant, or a quarter of the meridian through Paris, should be taken as the standard unit of length, that a unit of capacity and a unit of weight be derived from it, and that the scale be uniform.

The report was accepted, and the primary unit was named the *meter*.

Two Frenchmen — Delambre and Mechain, practical surveyors — were appointed to measure an arc of the meridian from Dunkirk, France, to Barcelona, Spain. This arc was about 10 degrees long. From this, they calculated the length of the quadrant, and took a ten-millionth part of it for the meter — whence the phrase "The Metric System."

This unit is very close to 39.37 inches in length. Consider the circumference of the earth to be 24855.0694 miles in length, which reduced to inches $= 24855.0694 \times 5280 \times 12$ inches. Then the length of the quadrant $= \frac{24855.0694 \times 5280 \times 12}{4}$ inches, and the ten-millionth part of it $= \frac{24855.0694 \times 5280 \times 12}{4 \times 10000000}$ inches $= 39.37043$ inches, which is approximately correct. This result was obtained by Clark in 1868, and while subsequent trials have shown that this result is slightly in error, it has been retained. The result of the committee's report was *adopted* by France in 1840, and made *legal* in the United States in 1868.

The multiples and submultiples of the meter and of the units derived from it are based on the decimal scale.

The multiple units are designated by prefixes derived from the Greek. They are: deka, 10; hecto, 100; kilo, 1000; myria, 10,000; mega, 1,000,000; etc. The sub-

multiple units are designated by prefixes derived from the Latin. They are: deci, $\frac{1}{10}$; centi, $\frac{1}{100}$; milli, $\frac{1}{1000}$; micro, $\frac{1}{1000000}$; etc.

Illustrations of both kinds of prefixes are found in decade, decimal, centennial, millennium, hecatomb, myriad, etc., which are common words.

Table of Linear Units

10 millimeters (mm.)	= 1 centimeter (cm.)
10 cm.	= 1 decimeter (dm.)
10 dm.	= 1 meter (m.)
10 m.	= 1 dekameter (Dm.)
10 Dm.	= 1 hectometer (Hm.)
10 Hm.	= 1 kilometer (Km.)
10 Km.	= 1 myriameter (Mm.)

The radix is 10.

SURFACE UNITS

A surface unit is a portion of surface in the form of a square, each side of which is some linear unit in length. Hence, the radix is 100.

100 square millimeters (sq. mm.)	= 1 square centimeter (sq. cm.)
100 sq. cm.	= 1 square decimeter (sq. dm.)
100 sq. dm.	= 1 square meter (sq. m.)
100 sq. m.	= 1 square dekameter (sq. Dm.)
100 sq. Dm.	= 1 square hectometer (sq. Hm.)
100 sq. Hm.	= 1 square kilometer (sq. Km.)

UNITS OF VOLUME, OR CUBIC UNITS

In general, the units of volume are cubes, each edge of which is some linear unit in length. It is evident that a cube whose edge is 10 times as long as the edge of another

cube is 1000 times as large. Hence, the radix is 1000. Hence:

1000 cubic millimeters (cu. mm.)	= 1 cubic centimeter (cu. cm.)
1000 cu. cm.	= 1 cubic decimeter (cu. dm.)
1000 cu. dm.	= 1 cubic meter (cu. m.)

Larger units than the cubic meter would be in the same ratio, but there is not much use for larger ones.

For measuring wood, the cubic meter is the unit, and is called a *stere* (st.).

The radix is 10. Make a table for wood measure.

UNITS OF CAPACITY

The primary unit is the *liter*. The *liter* is defined as equal in capacity to a cubic decimeter. The radix is 10.

10 milliliters (ml.)	= 1 centiliter (cl.)
10 cl.	= 1 deciliter (dl.)
10 dl.	= 1 liter (l.) = 1 cu. dm.
10 l.	= 1 dekaliter (Dl.)
10 Dl.	= 1 hectoliter (Hl.)
10 Hl.	= 1 kiloliter (Kl.)

1 Kl. = 1 cu. m.; hence is rarely used as a unit.

UNITS OF WEIGHT

The primary unit of weight is the *gram*. The *gram* is defined as the weight of 1 cu. cm. of water at its greatest density. With the proper apparatus, it is convenient to weigh with the metric weights 1 cu. dm. of water (1000 cu. cm.) to manifest the definition. The radix is 10.

10 milligrams (mg.)	=	1 centigram (cg.)
10 cg.	=	1 decigram (dg.)
10 dg.	=	1 gram (g.)
10 g.	=	1 dekagram (Dg.)
10 Dg.	=	1 hectogram (Hg.)
10 Hg.	=	1 kilogram (Kg.)
10 Kg.	=	1 myriagram (Mg.)
10 Mg.	=	1 quintal (Q.)
10 Q.	=	1 tonneau (T.)

Kilogram is usually written *kilo*, and 1 cu. m. of water weighs 1 metric ton.

For land measure, a special surface is used, called the *are*. The radix is 10. Hence:

10 centares (ca.)	=	1 deciare (da.)
10 da.	=	1 are (a.)
10 ar.	=	1 dekare (Da.)
10 Da.	=	1 hectare (Ha.)

The *are* is defined as 1 sq. Dm.

APPARATUS FOR SCHOOL USE

1. One meter stick which shows 1 dm., 1 cm., and 1 mm.
2. One tape 30 or 40 m. long, showing decimeters and centimeters.
3. One nest of liter units from 1 cl. to 5 l. inclusive.
4. One bed of metric weights from 1 g. to 1 Kg. inclusive.
5. One graduated liter cup and one dissected cubic decimeter.

REMARKS CONCERNING UNITS

Although the meter is derived from the arc of a meridian, it is a singular fact that neither it nor any part of it nor any multiple of it is used in measuring circles. The old units are preserved.

In any system of measurement, if the primary unit is too large, there is difficulty in apprehending the exact value of any magnitude; and all quantities which are smaller than the unit must be expressed fractionally. If the primary unit of the English system, the yard, were the smallest unit of it, the great majority of the numbers obtained in the practical world would be fractional.

Again, if the primary unit were too small, numbers would be inconveniently large. It is easy to comprehend the mile, but a mile expressed in inches would be hard to comprehend.

Hence, a system of weights and measures should have a sufficient range of units as to size to avoid the excessive use of fractions on the one hand and of large numbers on the other.

REDUCTION

A metric number of any denomination may be reduced to another denomination by simply shifting the decimal point.

In reducing from a higher to a lower denomination, move the decimal point to the right as many places as there are intervals between the given denomination and the required one. In reducing from a lower to a higher denomination, move the decimal point to the left as many places as there are intervals between the given and the required denomination.

EXERCISES

1. Reduce 1753 m. to Hm.; to Mm.; to dm.; to mm.
2. Reduce 26.5 m. to cm.; to Dm.; to Km.; to Hm.
3. Reduce .17 Hm. to Km.; to cm.; to Mm.; to m.
4. Reduce 2003 Km. to Dm.; to cm.; to Hm.; to m.
5. Reduce 28.3 l. to cl.; to Dl.; to Hl.; to Kl.
6. Reduce 2680 cl. to l.; to Kl.; to ml.; to Hl.
7. Reduce 6475 g. to Dg.; to dg.; to Kg.; to cg.
8. Reduce 84.326 Kg. to Dg.; to g.; to dg.; to Mg.

EXERCISES

1. Reduce 375.4 a. to Da.; to da.; to sq. Dm.; to sq. Hm.
2. Reduce 41.62 Ha. to sq. m.; to sq. cm.; to a.; to Da.
3. Reduce 293,465 sq. mm. to sq. dm.; to sq. Dm.; to a.
4. Reduce .743 cu. m. to cu. dm.; to cu. mm.; to cu. Dm.
5. Reduce 17,341 cu. cm. to cu. dm.; to cu. mm.; to cu. Dm.
6. Reduce .784 cu. m. to cu. dm.; to l.; to cu. Dm.; to dl.
7. Reduce 298,000 cu. cm. to l.; to Kl.; to Dl.; to dl.
8. Reduce .715 Hl. to cu. m.; to cu. cm.; to cu. Dm.

EXERCISES

1. How many m. in 93 Km.? in 274 Hm.? in 2873.5 cm.?
2. How many m. in the sum of 318 m., 18.52 Hm., and 600.3 Dm.?
3. How many Hl. in the sum of 892 Dl., 2030 l., .002 Ml., 2.18 Kl., and 296.65 Ml.?
4. From the sum of 26 Kg., .031 Mg., and 341 Dg. take 21,800 cg.
5. What is the weight in Kg. of 342.5 cu. dm. of water? Of 29.73 cu. m.? Of .003 cu. Dm.?
6. How many dg. does a l. of water weigh?

7. How many Kg. does a cu. m. of water weigh?

8. In 82.34564 Kl. of water, how many cu. Dm.? How many l.? How many cu. mm.? It weighs how many dg.? Dg.? Kg.? met. T.?

EXERCISES

1. 3.14 dm., 306.7 mm., 5219 Dm., and 1703 cm. are to be added. Express the sum in Dm.

2. 705 Mg. + 372 Dg. + 19 Kg. + 6 T. + 171 Mg. = how many Kg.?

3. 28.47 cg. × 14,930 = how many Kg.?

4. Divide 4036.25 dm. by .125, and express in m.

5. Divide 3600 Km. by 40 Dm.; by 72 m.; and by 45 cm.

6. Iron is 7.21 times as heavy as water. What will a cubic block of iron weigh in Kg., if one edge is 8 dm. long?

7. Express .36752 cl. of water as cu. mm., and find its weight in cg.

8. 43.7 mg. × 8.45 = what number of Kg.?

9. Divide 7.43244 dg. by 1.446 cg.

10. In a board 6 m. long and 4 dm. wide, how many sq. dm.?

11. A board walk is 640 m. long and 6.4 Dm. wide. How many ca. are there in it?

12. How many tiles, each 6 dm. by 4 dm., are required to cover a floor 32 m. by 180 dm.?

13. In 26.7 Ds. of wood, how many ds.? How many st.? How many cu. dm?

14. How many sq. Km. in 9,128,000 sq. dm.? How many sq. dm. in .0009 Ha.?

EXERCISES

1. A given volume of iron weighs 7.21 times as much as the same volume of water. This is expressed by saying that the *specific gravity* of iron is 7.21. What is the weight in grains of cu. dm. of iron?

2. The specific gravity (sp. gr.) of gold is 18.5. Find the weight of a cu. cm. of gold in dg.

3. The sp. gr. of platinum is 23. Find the weight in cg. of a rectangular piece of platinum which is 15 cm. long, 1.2 cm. wide, and 8 mm. thick.

4. Find the weight in dg. of 3 cu. cm. of copper, sp. gr. = 8.8.
Find the weight in cg. of 16 dl. of oil, sp. gr. = .9.

5. What will it cost to excavate a cellar 1.6 Dm. long, 12 m. wide, and 18 dm. deep, at 50¢ per cu. m.?

6. How many liters in a box, rectangular in form, 2.75 m. long, 16.5 dm. wide, and 124 cm. deep?

7. Find the cost of a pile of wood 175 dm. long, 1.6 m. wide, and 216 cm. high, at $2.50 per stere.

8. If the sp. gr. of a liquid is 1.5, how many grams of it will fill a cubical vessel whose edge is 1.8 m. long?

9. How long must a box be that is 15 dm. wide and .12 Dm. deep to hold 5400 l.?

10. A cistern is 3 m. long, 22 dm. wide, and 125 cm. deep. If 1.64 Hl. of water run out per minute, how long will it take to empty the cistern?

11. The nickel 5-cent piece weighs 5 grams. What is the weight in Dg. of $14 worth of such coins?

(The 5-cent piece is $\frac{3}{4}$ copper and $\frac{1}{4}$ nickel.)

Table of Equivalents

1 meter	= 39.37 in. = 3.28 ft. = 1.09 yd. (approx.).
1 Km.	= $\frac{5}{8}$ mi. (approx.).
1 are	= $\frac{1}{40}$ of an acre (approx.).
1 stere	= 1.308 cu. yd. (approx.).
1 liter	= 1.0567 liquid qt. = .908 dry qt.
1 gram	= 15.432 gr. troy.
1 Kg.	= 2.204 lb. avoirdupois, or $2\frac{1}{5}$ lb. (approx.).

EXERCISES

1. Express 14 Hm. in yards.

SOLUTION. 14 Hm. = 1400 m. But 1 m. = 1.0936 yd.
∴ 14 Hm. = 1400 times 1.0936 yd. = 15.3104 yd.

2. Express 23 Km. in miles; in rods; in yards.

3. From Terre Haute to Evansville it is 109 miles. How many Km.?

4. Express 36 Dm. in chains.

5. In 296 Ha. how many acres?

6. In 294 dl. how many cubic inches?

7. In 75.6 st. how many cords?

8. In 1 Kl. how many cubic yards?

9. In 8.213 Hl. how many cubic feet?

10. In 8.42265 ml. how many pints?

11. In 1 ca. how many square feet?

12. In 81.54 ds. how many cubic feet?

13. In 731 cg. how many pints?

14. In .075 T. (metric) how many hundredweight?

15. In 2.2 Ha. how many acres and lower denominations?

16. In 722.624 sq. dm. how many square yards?

17. Reduce 2523.1344 m. to a compound number in English System.

18. If one's weight is 50.24 Kg., what is it in pounds?

EXERCISES

1. If 25 Ha. of land, purchased at $864 per Ha., is sold at 10 cents per sq. m., find the gain.

SOLUTION. 25 Ha. at $864 will cost $864 × 25 = $21600. 25 Ha. = 2500 a. = 2500 sq. Dm. = 250000 sq. m. 250000 sq. m. sold at 10¢ per sq. m. sells for $25000. $25000 − $21600 = $3400. ∴ $3400 was gained.

2. If the above land had been sold at $400 per acre, how much would have been gained?

3. If I buy 24.625 Hl. of corn at $22.50 per Kl. and sell it at $2 per cu. dm., do I gain or lose?

4. If the sp. gr. of sulphuric acid is 1.8, what is the weight of 7.5 Dl. of the acid? Answer in dg.

5. How many Ha. of land in 75 sq. ch.?

6. How many acres in a plot of ground 40 Dm. long, 250 m. wide?

7. A pile of wood is 4.5 m. long, 1.4 m. wide, and 1.8 m. high. What is it worth at $5.75 per cord?

8. The latitude of one city is 45° 31′, and that of another is 42° 39′ 50″ North latitude. If they are on the same meridian, how many Km. are they apart?

9. Find the value of a pile of wood 20 ft. long, 5 ft. wide, and 6 ft. high, at $1.80 per stere.

10. A man bought 30 m. of cloth at $1.80 per m., and sold it at a profit of $25. At what price per yard was it sold?

11. A field is 50 Hm. long and 3000 dm. wide. How many ares in it? How many acres?

12. The distance between two places on a map is 175 mm. If the scale is 1 to 50,000, find the real distance in Km.

13. If I buy 500 bu. wheat at 76¢ per bushel, and sell it at $3 per Hl., do I gain or lose?

14. If a piece of land 6 Dm. square cost $150, at what price per are must it be sold to gain $75?

15. A rectangular box is 6 m. long, 32 dm. wide, and 240 cm. deep. How many dl. will it hold? How many cu. m.? How many Kg. of water will it hold? How many gallons?

16. If a man walks at the rate of 4 mi. per hour, how many hours and minutes will it take him to walk 7.5 Km.?

17. If it costs $6 to travel 450 mi. by rail, what is the rate in fare per mile?

18. Two places are on the equator, and the longitude of the one is 62° 30′ 20″ W. and that of the other is 36° 20′ 40″ W. How many Km. between them?

MISCELLANEOUS EXERCISES

1. The 5-cent nickel piece is 20 mm. in diameter. How many laid side by side will reach 2020 Dm.?

2. A copper cent is $\frac{9}{7}$ of a mm. thick. How many dollars' worth of such coins must be piled to reach to the height of 1 Km. 7 Hm. 3 Dm. 5 m.?

3. How many Km. per hour does a person walk who takes 5 steps in 2 seconds and 1500 steps in a Km.?

4. A boy hauls in $\frac{5}{36}$ of his kite-string, when he observes that 93 m. remain. How long is the kite-string?

5. The silver quarter-dollar is 24 mm. in diameter. $240,648 worth of such coins will extend how many Km.?

6. If each section of a drain tile is 8 dm. long, how many sections are required to lay a drain 7 Km. in length?

7. A bale of cotton weighs 500 lb. One half-pound will make a spool of thread containing 37.2 Dm. of thread. If a bale of cotton cost 16¢ per pound, find the profit in a bale of cotton made into thread and sold at 75¢ per Km.

8. A rectangular field is 40 rd. long and 28 rd. wide. If fence-posts are $2\frac{1}{2}$ m. apart, how many posts will be required to fence it?

9. If a carpet is 8.4 dm. wide, how many meters of carpet will be required to cover a floor 5 m. 6 dm. long and 4.2 m. wide?

10. A street is 18.4 m. wide and 12.45 Hm. long. What will it cost to pave it at 28 cents per sq. m.?

11. If fence-posts are set 2 m. apart, how many are required to fence in a rectangular farm containing 5 Ha.?

12. A field is twice as long as wide, and contains 36 Ha. of land. How many rods in its perimeter?

13. A lot is 20 rd. long. How wide must it be to contain 40 a.?

14. A rectangular field has a walk around it $8\frac{1}{4}$ ft. wide; the outer edge of the walk is 16 ch. long on one side of the field and 12 ch. on the other side. How many ares of land in the field?

15. The bottom of a rectangular cistern contains 10 sq. m. How deep must it be to hold 15,000 l.?

16. A piece of land contains 12 Ha., and there is a tree for each 6 sq. m. of land, which tree makes $4\frac{1}{2}$ dst. of wood. What is the timber worth at $3 per stere?

17. If 6 shot are made from 1 cu. cm. of lead, how many shot may be made from 1 bar, 3 dm. long, 20 cm. wide, and 1.5 cm. thick?

18. If the Central Union Telephone Co. has 6 direct wires from Indianapolis to Richmond, a distance of 69 mi., and the wires are of such size that 48 m. of it can be made from 1 cu. dm. of metal, how many cu. dm. in the 6 wires?

19. A box 3 m. 6 dm. 3 cm. long, 24 dm. 5 cm. wide, 15 dm. 2 cm. deep, will contain how many bushels?

20. 4.5 Kl. less 4.2 cu. m. equals how many cu. dm.?

21. How many Dm. in length must a trough be to hold 8 Kl. 7 Hl. 5 Dl., if the surface of the inside of one end contains $7\frac{1}{2}$ sq. dm.?

22. A crib is 12.5 m. long, 44 dm. wide, and 2.5 m. deep; how many dl. does it hold?

23. A tank which holds 45 gallons of wine holds how many Hl.?

24. A nickel 5-cent piece weighs 5 grams; and a silver half-dollar weighs $12\frac{1}{2}$ grams; what is the combined weight of $ 12 in nickels and $ 40 in half-dollars?

25. If a bar of steel weighing 5 Kg. be made into 3200 pens, find the weight of 1 pen in dg.

26. During a rainfall water fell to a depth of 3 cm. on the average over 45 Ha. How many metric tons fell?

27. If a mould is made with a silver half-dollar, and then it is filled with lead, how many grams does the lead weigh?

28. If a bed of coal covers 4 A. to the depth of 2 m., how many tons (2000 lbs.) can be mined from it?

29. A vessel full of milk weighs 45 Hg. and the empty vessel weighs 175 Dg. The vessel holds 267 cl.; find the sp. gr. of milk to two decimal places.

30. A train from London to Cambridge (90 Km.) is scheduled at 72 Km. per hour; when halfway it is 3 minutes late. At what rate must it travel the remainder of the journey to arrive on time?

31. The surface of a certain lake is 48 Ha. and the water is frozen so that the ice is 125 mm. thick. The sp. gr. of ice is .93. Find the weight of the ice on the lake in metric tons.

UNITS OF VALUE

Money is the universal medium of exchange of values. Money is often defined as the measure of the value of a thing. It is so called from the temple of Juno, Moneta, in which money was first coined in Rome.

Because money circulates through the country it is called the "currency" of a country, from *curro*, I run.

By the term "specie" is meant the coin of a country, or the metal money.

Paper currency is not money, but a promise to pay money, and circulates, or not, according to the strength of the security of the promise to pay. Metal money or specie is usually made from gold, silver, nickel, or copper, with the proper stamp placed upon it to make it legal.

UNITED STATES MONEY

The primary unit is the *dollar*. The word "dollar" is derived from the word *thal*, meaning a dale or valley.

The dollar was first coined in the first part of the sixteenth century in Joachim's valley in Bohemia. By some it is thought that the word "dollar" is from a Swedish word, *daler*.

The derived units are: the dime, one-tenth of a dollar, from the word *disme*, meaning one-tenth; the cent, one-

hundredth of a dollar, from *centum*, meaning one-hundredth; the mill, one-thousandth of a dollar, from *mille*, meaning one-thousandth; the eagle, ten dollars, from the eagle on the coin; the quarter-eagle; the half-eagle; the double-eagle.

10 mills	= 1 cent
10 cents	= 1 dime
10 dimes	= 1 dollar ($)
10 dollars	= 1 eagle

The scale is uniform and decimal; the words which express the units, however, do not give this regular decimal scale.

The coins are: the *cent* and *two-cent* coins (the latter not made now), made of copper; the *five-cent* coin, made of nickel; the *dime*, the *quarter*, the *half-dollar*, and the *dollar*, made of silver; the *quarter-eagle*, the *half-eagle*, the *eagle*, and the *double-eagle*, made of gold.

Gold dollars, silver three-cent pieces, and half-dimes of silver exist, yet are rare.

Copper coins are merely token money, not legal for large sums.

The symbol ($) is supposed to be the result of placing the letter U on the letter S in U.S.

ENGLISH MONEY

The primary unit is the *pound sterling*, made of gold. The derived units are: the shilling, made of silver; the penny, the farthing, both made of copper; the crown, half-crown, the florin, all made of silver; and the guinea and half-guinea, made of gold, but not coined any more.

4 farthings (far.)	= 1 penny (*d.*)	= $0.02
12 pence	= 1 shilling (*s.*)	= $0.2433
20 shillings	= 1 pound (£)	= $4.8665
21 shillings	= 1 guinea	= $5.11
2 shillings	= 1 florin	= $0.4866
2 shillings 6 pence	= 1 half-crown	= $0.608
5 shillings	= 1 crown	= $1.216

Farthing is supposed to mean a fourth of a thing.

FRENCH MONEY

The French system is founded on the decimal notation. The primary unit is the *franc*, and the derived units are the decime, centime, and millime. The millime, centime, and decime are coins made of silver; the franc and the five-franc are made of silver or gold; the ten-franc, the twenty-franc, and the forty-franc coins are made of gold.

10 millimes = 1 centime
10 centimes = 1 decime
10 decimes = 1 franc = $0.193 = 19.3 cents

How should the *five-franc* silver coin compare in size with the United States dollar?

GERMAN MONEY

The primary unit is the *reichsmárk*, or, in short, the *mark*, equal in value to $0.2385 = 23.85 cents. The mark is divided into 100 parts, each part called a *pfennig*, equal in value to about $\frac{1}{4}$ cent.

SPANISH MONEY

The primary unit is the *peseta*, made of silver, and equal to $0.19̶3, or 19.3 cents. The coin of 5 pesetas is made of silver, and the coin of 25 pesetas is made of gold.

"One Faith, one weight, one measure, and one coin
Would all the world in harmony conjoin."
— O'Sullivan's *Arithmetic.*

FOREIGN MONEY VALUES

Value of Coins July 1, 1899

Countries	Standard	Monetary Unit	Value in U.S. Gold Dollar	Coins
Argentine Republic	Gold and silver	Peso	$0.965	Silver: peso and divisions Gold: ½ and 1 Argentine
Austria-Hungary	Gold	Crown	$0.203	Gold: 1, 10, and 20 crowns
Belgium	Gold and silver	Franc	$0.193	Silver: franc and 5-franc piece Gold: 10 francs and 20 francs
Bolivia	Silver	Boliviano	$0.443	Silver
Brazil	Gold	Milreis	$0.546	Silver: ½, 1, and 2 milreis Gold: 5, 10, and 20 milreis
British Honduras	Gold	Dollar	$1.000	
Chili	Gold	Peso	$0.365	Silver: peso and divisions Gold: escudo, $1.875; doubloon, $3.650; condor, $7.300
China	Silver	Tael	$0.714	The value varies in different sections.
Denmark	Gold	Crown	$0.268	Gold: 1 crown, 10 and 20 crowns
Egypt	Gold	Pound	$4.943	Silver: 1, 2, 5, 10, and 20 piasters Gold: 5, 10, 20, and 50 piasters
France	Gold and silver	Franc	$0.193	Silver: 1 and 5 francs Gold: 5, 10, 20, 50, and 100 francs

FOREIGN MONEY VALUES — *Continued*

Countries	Standard	Monetary Unit	Value in U.S. Gold Dollar	Coins
German Empire	Gold	Mark	$0.238	Gold: 5, 10, and 20 marks
Great Britain	Gold	Pound sterling	$4.866	Silver: shilling Gold: pound
Greece	Gold and silver	Drachma	$0.193	Silver: 1 and 5 drachmas Gold: 5, 10, 20, 50, and 100 drachmas
Italy	Gold and silver	Lira	$0.193	Silver: 1 and 5 lire Gold: 5, 10, 20, 50, and 100 lire
Japan	Gold	Yen	$0.498	Silver: 10, 20, and 50 sen Gold: 5, 10, and 20 yen
Mexico	Silver	Dollar	$1.014	Silver: peso and divisions Gold: 1, 2½, 5, 10, and 20 dollars
Norway	Gold	Crown	$0.268	Gold: 10 and 20 crowns
Persia	Silver	Kran	$0.082	Silver: ¼, ½, 1, 2, and 5 krans Gold: ½, 1, and 2 tomans
Peru	Silver	Sol	$0.443	Silver: sol and divisions
Russia	Gold	Ruble	$0.515	Silver: ¼, ½, and 1 ruble Gold: 7½ and 15 rubles
Spain	Gold and silver	Peseta	$0.193	Gold: 25 pesetas Silver: 1 and 5 pesetas
Switzerland	Gold and silver	Franc	$0.193	Same as France
Turkey	Gold	Piaster	$0.044	Gold: 25, 50, 100, 250, and 500 piasters

UNITS OF TIME

Units of time are natural, being definite portions of duration fixed by the rotation of the earth on its axis, and its revolution around the sun.

The primary unit is the day, which is determined by the rotation of the earth on its axis.

The derived units are the second, minute, hour, week, month, year, and century.

The second and the minute get their derivation from the corresponding divisions of the circle into units of the same name.

The word "hour" is from a Latin word, *hora*, meaning a definite portion of time.

The week, as a unit of time, is supposed to be derived from a traditional account of the creation of the universe in six days, the seventh being given to rest.

The month is the period of time in which the moon makes its revolution around the earth. Our names for the months are borrowed from the Romans.

January is derived from Janus, the name of an old Italian deity, to whom all beginnings were sacred.

February comes from *februare*, to expiate, and was originally the month of expiation.

March is from Mars, the god of war, in Roman mythology.

April is from a Latin word, *aperire*, which means to open, and is the month of the opening of buds and the earth for producing growth.

May is from Maia, the mother of Mercury, to whom the Romans offered sacrifices on the first day of the month.

June is from Juno, the sister of the wife of Jupiter.

July is from Julius, in the name of Julius Cæsar, who was born in this month.

August is from Augustus, in the name of Augustus Cæsar, who was the first emperor of Rome.

September, October, November, and December are derived from Latin numerals, and mean the seventh, eighth, ninth, and tenth months, respectively, counting from March.

The *true* year is equal to 365 days, 5 hours, 48 minutes, and 49.7 seconds.

The *common* year is equal to 365 days, which is 5 hours 48 minutes 49.7 seconds shorter than the *true* year. To compensate for this, an extra day is allowed in four years, which day is added to the 28 days of February, making 29 days for that month and 366 days for the year.

But, 4 times 5 hr. 48 min. 49.7 sec. = 23 hr. 15 min. 18.8 sec., which is 44 min. 41.2 sec. short of 24 hr. or 1 day. Hence, it is not exactly correct to add a day of 24 hr. in each four years. In 100 years, this 44 min. 41.2 sec. each four years, would amount to 25 times 44 min. 41.2 sec. = 18 hr. 37 min. 10 sec. Hence, once in 100 years the extra day in February is not counted. This is why the year 1900 was not a *leap year*.

But 18 hr. 37 min. 10 sec. is 5 hr. 22 min. 50 sec. short of a day of 24 hr. Four times this period is 21 hr. 31 min. 20 sec. Therefore in each 400 years a day *is* added, which corrects the calendar every 400 years to within 2 hr. 28

min. 40 sec. Hence, the calendar is corrected in something less than 4000 years as follows:

In general, count 365 days to the year. Every fourth year add 1 day extra, making 366 days to that year, called a *leap year*, except the centennial years, when a *leap year* is counted in each fourth *century* only.

The Gregorian calendar is the one now in use (see any good encyclopædia).

Centuries, years, months, weeks, days, hours, minutes, and seconds are reckoned from the beginning of the Christian era, beginning at midnight.

60 seconds (sec.)	= 1 minute (min.)
60 minutes	= 1 hour (hr.)
24 hours	= 1 day (da.)
7 days	= 1 week (wk.)
365 days	= 1 common year (52 wk. 1 da.)
366 days	= 1 leap year (yr.)
12 months (mo.)	= 1 year (yr.)
100 years	= 1 century (C.)

UNITS OF ANGULAR MEASURE

There are three methods of measuring angles.

I. The *sexagesimal* method, in which the primary unit is the degree (°), and the derived units are the minute (′) and the second (″).

II. The *centesimal* method, in which the primary unit is the grade (gr.), and the derived units are the minute (\`) and the second (\`\`).

III. The *circular* method, in which the unit is the *radian*. This is called the natural method.

The first and third methods are in common use.

The third method is used in advanced mathematics.

60 seconds (″)	= 1 minute (′)
60 minutes	= 1 degree (°)
360 degrees	= 1 circle

100 seconds (‶)	= 1 minute (‵)
100 minutes	= 1 grade (gr.)
100 grades	= 1 quadrant of 90°

The *degree* is the 360th part of a circle, or 90th part of a quadrant.

The *grade* is the 100th part of a quadrant of 90°.

The *radian* is the angle at the center of a circle whose arc is the length of the radius of the circle. The *centesimal* is the French method.

EXERCISES

1. In 1°, how many grades?
2. In 1′, how many minutes (‵)?
3. In 1″, how many seconds (‶)?
4. 1 gr. equals how many degrees?
5. 1‵ equals how many minutes (′)?
6. 1‶ equals how many seconds (″)?
7. Which is the larger, an angle of 4‵ or of 4′?
8. Reduce 24° 15′ 20″ to centesimal measure.
9. Reduce 15 gr. 10‵ 14‶ to common measure.
10. How many degrees in a radian?

Miscellaneous Tables

12 things	= 1 dozen (doz.)	24 sheets	= 1 quire
12 dozen	= 1 gross	20 quires	= 1 ream
12 gross	= 1 great gross	2 reams	= 1 bundle
20 things	= 1 score	5 bundles	= 1 bale

A *sheet* of anything is a broad, thin portion of that thing: *e.g.* a sheet of paper, or a sheet of tin.

A sheet folded in 2 leaves is called a folio.
A sheet folded in 4 leaves is called a quarto (4to).
A sheet folded in 8 leaves is called an octavo (8vo).
A sheet folded in 12 leaves is called a 12mo.
A sheet folded in 16 leaves is called a 16mo.
A sheet folded in 24 leaves is called a 24mo.

50 lb. green apples make 1 bu.
24 lb. dried apples make 1 bu.
48 lb. barley make 1 bu.
60 lb. white beans make 1 bu.
14 lb. blue grass seed make 1 bu.
52 lb. buckwheat make 1 bu.
60 lb. clover seed make 1 bu.
70 lb. corn in ear make 1 bu.
56 lb. shelled corn make 1 bu.
56 lb. flaxseed make 1 bu.
32 lb. oats make 1 bu.
57 lb. onions make 1 bu.
60 lb. Irish potatoes make 1 bu.
55 lb. sweet potatoes make 1 bu.
56 lb. rye make 1 bu.
45 lb. timothy make 1 bu.
60 lb. wheat make 1 bu.
200 lb. pork or beef make 1 bbl.
196 lb. flour make 1 bbl.
280 lb. salt make 1 bbl.
256 lb. soap make 1 bbl.
25 lb. powder make 1 keg.

RELATION OF NUMBER

1. 1 is what part of 2? Of 3? Of 4? Of 6?
2. 2 is what part of 4? Of 6? Of 8? Of 12?
3. 2 is what part of 3? Of 5? Of 9? Of 15?

Solution. 2 is $\frac{2}{5}$ of 5. 2 is $\frac{2}{9}$ of 9. 2 is $\frac{2}{15}$ of 15.

4. 3 is what part of 4? Of 10? Of 14? Of 19?

5. 4 is what part of 7? Of 11? Of 16? Of 101?

SOLUTION. Say 4 is $\frac{4}{11}$ of 11; *not* 1 is $\frac{1}{11}$ of 11, then 4 is $\frac{4}{11}$ of 11.

6. 10 is what part of 16? Of 29? Of 30?

7. If 6 oranges cost 15 cents, what will 18 oranges cost?

SOLUTION. Since 18 oranges cost 3 times as much as 6 oranges, 18 oranges will cost 3 times 15¢ = 45¢. (Do not find the cost of 1 orange.)

8. 7 pk. are what part of 14 pk.? Of 21 pk.? Of 32 pk?

9. 19 apples are what part of 57 apples? Of 80 apples?

10. 54 desks are what part of 216 desks? Of 270 desks?

11. If 24 sheep cost $96.48, find the cost of 120 sheep.

SOLUTION. Since 120 sheep = 5 times 24 sheep, they will cost 5 times $96.48 = $482.40. (Do not find the cost of 1 sheep.)

12. If 25 lb. sugar cost $2, find the cost of 200 lb.

13. If 400 bu. corn cost $280, find the cost of 16 bu.

14. If 10 cakes of maple sugar cost 75¢, find the cost of $3\frac{1}{3}$ cakes.

SOLUTION. Since $3\frac{1}{3}$ cakes are $\frac{1}{3}$ of 10 cakes, they will cost $\frac{1}{3}$ of 75¢ = 25¢.

15. If $4\frac{1}{2}$ lb. pork cost 37¢, find the cost of $13\frac{1}{2}$ lb.

16. $\frac{1}{2}$ is what part of 5? Of 8? Of 54?

SOLUTION. The operation is the same as that in Prob. 1. Divide $\frac{1}{2}$ by 5; $\frac{1}{2} \div 5 = \frac{1}{10}$; $\therefore \frac{1}{2}$ is $\frac{1}{10}$ of 5.

17. $\frac{1}{3}$ is what part of 2? Of 7? Of 12? Of 19?

18. $\frac{2}{5}$ is what part of 6? Of 9? Of 16? Of 29?

SOLUTION. $\frac{2}{5} \div 6 = \frac{2}{30} = \frac{1}{15}$; $\therefore \frac{2}{5}$ is $\frac{1}{15}$ of 6.

19. $$\frac{8}{9}$ is what part of $12? Of $15? Of $20?

20. $2\frac{1}{2}$ pk. is what part of 5 pk.? Of $17\frac{1}{2}$ pk.?

21. $6\frac{2}{3}$ ft. is what part of 20 ft.? Of $53\frac{1}{3}$ ft.?

22. If $5\frac{1}{3}$ lb. steak cost 70¢, find the cost of 16 lb.

SOLUTION. 16 lb. = 3 times $5\frac{1}{3}$ lb.; then 16 lb. cost 3 times 70¢ = 210¢ = \$2.10. (Do not find cost of 1 lb.)

23. If $4\frac{2}{3}$ bu. corn cost \$2.80, find the cost of $2\frac{1}{3}$ bu.

24. If $12\frac{1}{2}$ reams of paper cost \$45, find the cost of $2\frac{1}{2}$ reams.

25. If $3\frac{1}{2}$ yd. cloth cost \$11, find the cost of 8 yd.

26. $\frac{2}{3}$ is what part of $\frac{4}{3}$? Of $\frac{8}{3}$? Of $\frac{14}{3}$?

27. $\frac{3}{4}$ is what part of $\frac{9}{4}$? Of $\frac{4}{5}$? Of $\frac{6}{7}$?

28. \$$\frac{8}{9}$ is what part of \$$\frac{16}{9}$? Of \$$\frac{9}{10}$? Of \$$\frac{7}{18}$?

SOLUTION. 8 shells are $\frac{1}{2}$ of 16 shells; 8 ninths are $\frac{1}{2}$ of 16 ninths; $\frac{8}{9} \div \frac{16}{9} = \frac{8}{16} = \frac{1}{2}$. $\therefore$ \$$\frac{8}{9}$ are $\frac{1}{2}$ of \$$\frac{16}{9}$.

29. $2\frac{2}{9}$ is what part of $3\frac{1}{3}$?

SOLUTION. $2\frac{2}{9}$ is such part of $3\frac{1}{3}$ as is shown by $\dfrac{2\frac{2}{9}}{3\frac{1}{3}}$. Now multiply both terms by 9. $\therefore \dfrac{2\frac{2}{9}}{3\frac{1}{3}} = \dfrac{2\frac{2}{9} \times 9}{3\frac{1}{3} \times 9} = \frac{20}{30} = \frac{2}{3}$. $\therefore 2\frac{2}{9}$ is $\frac{2}{3}$ of $3\frac{1}{3}$.

30. If $13\frac{2}{3}$ rd. of fence cost \$4.25, find the cost of $16\frac{2}{5}$ rd.

31. If 5 articles cost \$17, find the cost of 30 articles.

32. If 18 articles cost \$50, find the cost of 9 articles.

33. If 25 articles cost \$40, find the cost of 60 articles.

34. If it takes $4\frac{1}{2}$ hr. to travel 135 miles, how long will it take to travel 81 miles?

35. If 150 men earn \$200, what can 120 men earn in the same time?

36. If 12 horses draw 19 tons, how much can 18 horses draw?

37. If the interest on \$540 is \$27 per year, what is the interest on \$810?

38. If the interest on \$100 is \$$7\frac{1}{2}$ per year, what is the interest per year on \$14?

SOLUTION. \$14 is $\frac{14}{100}$ or $\frac{7}{50}$ of \$100. $\frac{7}{50}$ of \$7.50 = \$1.05. ∴ \$1.05 is the interest on \$14 for 1 yr.

39. If I pay \$22.50 for hauling 9 loads of wood, what will it cost to haul 8 loads?

40. If $7\frac{1}{2}$ yd. cloth cost \$40, find the cost of $22\frac{1}{2}$ yd.

41. How many knives can be purchased for \$280, if 3 knives cost \$1.20?

42. How many yards can be bought for 91¢, if 3 yd. cost 39¢?

43. How many pounds of candy can be bought for \$28, if 5 lb. cost \$3.50?

SOLUTION. \$28 is 8 times \$3.50; hence 8 times 5 lb., or 40 lb., can be bought for \$3.50.

44. If 78¢ purchase 13 yd. cloth, how many will \$1.17 purchase?

45. If \$$1.33\frac{1}{3}$ purchase 12 melons, how many will \$$6.66\frac{2}{3}$ purchase?

46. If $6\frac{1}{2}$ yd. silk cost \$19.50, how many yards can be bought for \$58.50?

47. If 42 men dig a ditch 75 rd. 4 ft. long, how many men can dig one 225 rd. 13 ft. 5 in. long in the same time?

48. If 30 men earn £5 18 *s.* 9 *d.*, how many men can earn £17 16 *s.* 3 *d.*?

49. If 5 bu. 6 pk. 3 qt. seed sow 12 A. 15 sq. rd., how much will 29 bu. 3 pk. 6 qt. sow?

50. If I can travel 80 mi. 100 rd. in 5 hr. 15 min., how long will it take me to travel 64 mi. 80 rd.?

51. How many times the distance that John can walk in $\frac{2}{3}$ hr. can he walk in 7 hr.?

52. If A owns $\frac{5}{6}$ of a farm and sells $\frac{3}{4}$ of his share, what part of the farm did he sell?

53. Thomas had $\$\frac{7}{10}$ and gave his sister $\$\frac{2}{5}$. What part of his money did he give his sister?

54. How much did Henry pay for 16 oranges at the rate of $2\frac{1}{2}$ oranges for $7\frac{1}{2}$¢?

55. The owner of $\frac{5}{6}$ of a lot sold $\frac{1}{2}$ of his share; what part of the lot did he sell?

56. If I expend $\$4\frac{1}{2}$ out of $\$5$, what part of the $\$5$ remains?

57. If $\frac{3}{4}$ of a pound of coffee cost 17¢, find the cost of 5 lb.

58. If 21 men require 69 da. to do some work, how long will 23 men take?

SOLUTION. They will take $\frac{21}{23}$ of 69 da. = 63 da.

59. How many yards worth 5¢ each must be exchanged for 40 yd. at $7\frac{1}{2}$¢ each?

60. If a sack of flour makes 28 loaves of bread weighing 2 lbs. each, how many 3-lb. loaves should it make?

61. If a gallons of vinegar cost c cents, how much will b gallons cost?

SOLUTION. Since b gallons $= \frac{b}{a}$ times a gallons, b gallons will cost $\frac{b}{a}$ times c cents $= \frac{bc}{a}$ cents.

RATIO

Ratio originates in the comparison of two numbers, or two quantities of the same kind. Ratio is the numerical relation between two numbers, or between two quantities of the same kind, and is determined by dividing the *first* by the *second*. Hence, the ratio of one quantity to another of the same kind is the *quotient* of the first by the second.

Thus, the ratio of 5 to 8 $= \frac{5}{8}$; of 3 bu. to 5 bu. $= \frac{3}{5}$. The ratio of 7 to 9 is usually expressed as $7 : 9$, where the symbol (:) is supposed to be the sign of division, $\div$, with the straight line left out. Hence, the ratio of 4 bu. to 5 bu. may be expressed 4 bu. : 5 bu. = 4 bu. $\div$ 5 bu. $= \frac{4}{5}$.

The *form* of expression $9 : 11$ is traditional and is good; yet the ratio of 9 to 11 should be expressed $\frac{9}{11}$.

The two numbers or quantities whose ratio is sought are called the *terms* of the ratio. The first is called the *antecedent*, and the second is called the *consequent*.

Only quantities of the same kind have a ratio. Thus, the ratio of 7 qt. to 10 qt. $= \frac{7}{10}$ or $7 : 10$. If the relation or ratio 7 qt. to 2 gal. is wanted, either 7 qt. must be changed to gallons, or 2 gal. must be changed to quarts. The latter is preferable; the ratio of 7 qt. to 8 qt. $= \frac{7}{8}$. However, 7 qt. $= 1\frac{3}{4}$ gal. Then $1\frac{3}{4}$ gal. : 2 gal. $= \dfrac{1\frac{3}{4}}{2} = \dfrac{1\frac{3}{4} \times 4}{2 \times 4} = \frac{7}{8}$.

When each term of a ratio is a single integral number, the ratio is said to be *simple*. Thus, $\frac{2}{3}$ and $7 : 9$ are *simple*

ratios. If each term has two or more simple numbers in it the ratio is called a *compound* ratio. Thus, $\begin{matrix}(3:5)\\(4:6)\end{matrix}$, $\frac{(3)}{(4)}:\frac{(5)}{(6)}$, and $3 \times 4 : 5 \times 6$ are compound ratios. If either term of a ratio is a fraction, the ratio is called a *complex* ratio. Thus, $3 : 4\frac{1}{2}$ and $\frac{3\frac{1}{2}}{5\frac{1}{3}}$ are *complex* ratios.

The ratio $\frac{5}{3}$ is the *reciprocal*, or *inverse*, of the ratio $\frac{3}{5}$.

Since the terms of a ratio bear the relation of dividend and divisor, both terms may be multiplied or divided by the same number without changing the ratio. Thus, $3 : 7 = 6 : 14 = 21 : 49$; and $16 : 48 = 8 : 24 = 1 : 3$.

EXERCISES

1. Reduce $\frac{1}{2} : 2$ to a simple ratio.

   $(\frac{1}{2} : 2 = \frac{\frac{1}{2} \times 2}{2 \times 2} = \frac{1}{4} = 1 : 4.)$

2. Reduce $\frac{2}{3} : 4$ to a simple ratio.

   $(\frac{2}{3} : 4 = \frac{\frac{2}{3} \times 3}{4 \times 3} = \frac{2}{12} = \frac{1}{6}. \quad \therefore \frac{2}{3} : 4 = 1 : 6.)$

3. Reduce $3\frac{1}{3} : 4\frac{2}{5}$ to a simple ratio.
4. Reduce $19\frac{1}{2} : 23\frac{1}{3}$ to a simple ratio.
5. Reduce $2\frac{1}{3} : 5\frac{1}{4}$ to a simple ratio.
6. Reduce $1\frac{3}{4} : \frac{6}{1\frac{1}{2}}$ to a simple ratio.
7. Reduce $\begin{matrix}(3:4)\\(7:6)\end{matrix}$ to a simple ratio.

   $(3 \times 7 : 4 \times 6 = 21 : 24 = 7 : 8.)$

8. Reduce $\begin{matrix}(19:25)\\(15:38)\end{matrix}$ to a simple ratio.

9. Reduce $\begin{matrix}(1\frac{2}{3} : 5) \\ (10 : 1\frac{3}{7})\end{matrix}$ to a simple ratio.

10. Find the simple ratio of 4 gal. 3 qt. to 3 gal. 2 qt. 1 pt. (Reduce each term to same denomination.)

11. Find the ratio of a floor 16 ft. long and 14 ft. wide to a floor 15 ft. long and 12 ft. wide.

12. Which is the greater, $2:3$ or $7:8$?

13. Which is the greater, $\$2\frac{1}{2} : \$5\frac{1}{6}$ or $3\frac{1}{4}$ ft. : $6\frac{7}{8}$ ft.?

14. The length of one line is 7.9 meters, and that of another 23.7 meters. Find the ratio of the first to the second. Of the second to the first.

15. If the radius of a circle is 7 ft. and the circumference of it is 44 ft., find the ratio of the circumference to the diameter.

16. Find the ratio of the area of a circle, whose radius is $10\frac{1}{2}$ inches, to the circumference.

PROPORTION

$\frac{6}{8} = \frac{3}{4}$; $\frac{6}{7} = \frac{12}{14}$; and $\frac{2}{3} = \frac{22}{33}$.

These statements might have been put this way: 6 : 8 = 3 : 4; 6 : 7 = 12 : 14; 2 : 3 = 22 : 33. Again, 6 qt. : 8 qt. = 3 bu. : 4 bu.; 8 pt. : 10 pt. = 12 lb. : 15 lb.; and 3 pk. : 8 pk. = 15 oz. : 40 oz. The latter is read, the ratio of 3 pk. to 8 pk. equals the ratio of 15 oz. to 40 oz.

When two ratios are equal, they are said to form a *proportion*. Any two *like* quantities have a ratio, but not *every* two ratios form a proportion. The ratios must be equal.

There are two ratios and four terms in every proportion.

Four quantities are said to form a *proportion* when the ratio between two of them is equal to the ratio between the other two. Thus, 3 pt., 5 pt., 9 pt., and 15 pt. are four quantities such that the ratio of 3 pt. to 5 pt. equals the ratio of 9 pt. to 15 pt. 3 pt. : 5 pt. = 9 pt. : 15 pt.; or 3 pt. : 9 pt. = 5 pt. : 15 pt. is another proportion from the same four quantities.

Instead of the sign =, the double colon may be used. Thus, 5 pt. : 3 pt. : : 15 pt. : 9 pt.

The history of the use of the double colon (: :) is not very clear. While it is proper to use it, it is much better to use the equality sign (=).

The first and fourth terms of a proportion are called the *extremes*, and the second and third terms are called the *means*.

If the two ratios of a proportion are simple, the proportion is *simple*.

If either or both ratios of a proportion are compound, the proportion is *compound*.

FUNDAMENTAL PROPERTIES

When four numbers or quantities of the same kind are in proportion:

I. The product of the means equals the product of the extremes.

II. The ratio of the first and third terms of a proportion equals the ratio of the second and fourth terms.

III. The ratio of the second term to the first equals the ratio of the fourth term to the third.

IV. The ratio of the sum of the first and second terms to the first or second term equals the ratio of the sum of the third and fourth terms to the third or fourth term.

V. The ratio of the difference between the first and second terms to the first or second equals the ratio of the difference between the third and fourth terms to the third or fourth.

VI. All the terms of a proportion may be multiplied or all may be divided by the same number without destroying the equality of the ratios.

VII. All the terms of a proportion may be raised to the same power, or the same root of all the terms may be extracted, without destroying the equality of the ratios.

Illustrations of the principles stated above:

$3 : 6 = 8 : 16$. This is a true proportion by the definition of a proportion.

Hence, by I, $3 \times 16 = 8 \times 6$.

By II, $3:8=6:16$.

By III, $6:3=16:8$.

By IV, $3+6:3=8+16:8$; or, $3+6:6=8+16:16$.

By V, $6-3:3=16-8:8$; or, $6-3:6=16-8:16$.

By VI, $3\times 5:6\times 5=8\times 5:16\times 5$; or, $\frac{3}{5}:\frac{6}{5}=\frac{8}{5}:\frac{16}{5}$.

By VII, $3^2:6^2=8^2:16^2$; or, $\sqrt{3}:\sqrt{6}=\sqrt{8}:\sqrt{16}$. In general, $3^n:6^n=8^n:16^n$; or, $\sqrt[n]{3}:\sqrt[n]{6}=\sqrt[n]{8}:\sqrt[n]{16}$.

From the first fundamental property above, it follows that either extreme term of a proportion equals the product of the means divided by the other extreme term.

Thus, when $3:5=9:15$, the product of the means $=$ $9\times 5=45$. $45\div 3=15$; or, $45\div 15=3$.

It also follows that the product of the extremes divided by either mean term equals the other mean. From above, $15\times 3=45$. $45\div 9=5$; or, $45\div 5=9$.

Therefore, when any *three* terms of a proportion are given, the fourth term may be found, or the fourth term is determined. The method of finding the fourth term of a proportion when three are given used to be called the "single rule of three."

In the proportion $3:9=9:27$, 27 is a *third proportional* to 3 and 9; and 9 is a *mean proportional* between 3 and 27.

EXERCISES

Find the missing terms in the following:

1. $7:14=21:(\quad)$. $\qquad 9:15=(\quad):20$.
2. $6:(\quad)=5:10$. $\qquad (\quad):8=15:12$.
3. $42:(\quad)=72:50$. $\qquad 35:80=(\quad):16$.
4. $(\quad):108=25:30$. $\qquad 2\frac{1}{2}:(\quad)=8\frac{4}{7}:12\frac{3}{14}$.

It is convenient to place x for the missing term.

5. $5\% : 8\% = 7\% : x$. $8\% : 3\% = 12$ bu. $: x$.

6. 18 pt. $: x = 5 : 15$. 25 pk. : 40 pk. $= x :$ \$32.

7. $x : 75$ bbl. $= 1 : 3$. $46¢ : 69¢ = x : 12$ yd.

8. If 18 bbl. flour feed a company 8 wk., how long will 54 bbl. feed them?

SOLUTION. 18 bbl. : 54 bbl. = 8 wk. : x. Whence $x = \frac{54 \times 8}{18} = 24$. If 54 bbl. be put for the first term, then 54 bbl. : 18 bbl. $= x : 8$ wk. Whence $x = \frac{54 \times 8}{18} = 24$. If x be placed for the first term, then $x : 8$ wk. = 54 bbl. : 18 bbl. If 8 wk. be placed for the first term, then 8 wk. : x = 18 bbl. : 54 bbl. Whence $x = \frac{54 \times 8}{18} = 24$. Therefore, in any instance the result is 24 wk.

9. If 36 Ha. of land rent for \$64, find the rent of 80 Ha. (From this problem make the statements of the four proportions corresponding to the above.)

10. If 4 sheep cost \$17, find the cost of 219 sheep.

11. If \$840 draw \$46.20 interest, what will \$1260 draw in the same time?

12. What is the cost of 60.5 tons of coal when .9 ton costs \$6.60?

13. If $\frac{5}{9}$ of a warehouse is worth \$7000, what is $\frac{7}{12}$ of it worth?

14. If a train runs 180 Km. in 4 hr., how long will it take to run 100 Km.?

15. If 10.5 st. of wood cost \$12.50, find the cost of 41 cu. m. of the same kind of wood.

16. If 120 Hl. of oats feed some horses 9 days, how long will 225 Hl. feed them?

17. If a man walk 75 rd. in 3 min., how many hours will it take him to walk 12 mi.?

(3 min. : x = 75 rd. : 320 rd. $\times$ 12. $\therefore x = 153\frac{3}{5}$ min. $153\frac{3}{5}$ min. $= 2\frac{14}{25}$ hr.)

18. If 40.5 tons of hay cost \$810, find the cost of 51 tons.

19. If 500 bbl. apples cost \$2800, find the cost of 425 bbl.

20. If a vessel sail 2400 Km. in 18 days, how far will it sail in 48 days?

21. Find a mean proportional between 9 and 49.

22. Find a mean proportional between .16 and 5^2.

23. Find a mean proportional between .81 and 1.69.

24. Find a mean proportional between $.8^2$ and .0036.

25. Find a third proportional to 3.6 and 4.5.

26. Find a third proportional to .5 and 15.

27. Find a third proportional to 7^2 and $.35^2$.

28. Find a third proportional to .2 and 1.2.

COMPOUND PROPORTION

All of the problems usually solved by the method of *compound proportion* are much more easily solved by the method of *simple proportion*.

EXERCISES

1. If 6 men in 10 da., working 9 hr. per day, dig a ditch 150 rd. long, 2 ft. wide, and 3 ft. deep, how many men in 12 da. of 8 hr. each can dig one 200 rd. long, 6 ft. wide, and 4 ft. deep?

SOLUTION. 6 men in 10 da. of 9 hr. : x men in 12 da. of 8 hr. each = 150 rd. by 3 ft. by 2 ft. : 200 rd. by 4 ft. by 6 ft.

Whence $x = \dfrac{200 \times 4 \times 6 \times 6 \times 10 \times 9}{150 \times 3 \times 2 \times 12 \times 8} = 30.$ $\therefore$ 30 men.

2. If 9 men in 28 days build $247\frac{2}{13}$ rd. of wall, in how many days will 8 men build 51 rd.?

3. If $2\frac{1}{2}$ acres of pasturage support 5 horses for $3\frac{1}{2}$ days, how many acres would be required to support 26 horses for $17\frac{1}{2}$ days?

4. If it takes 35 Kg. of wool to make a piece of cloth 25 m. long, $\frac{3}{5}$ m. wide, how long a piece of cloth $\frac{1}{5}$ m. wide can be made from 112 Kg.?

5. If 5 pumps, each having a length of stroke of 3 ft., working $12\frac{1}{2}$ hr. a day for 6 da., empty a mine, what must be the stroke of each of 15 pumps which empty the same mine in 15 da. of 8 hr. each, the stroke of the former set of pumps being 4 times as fast as that of the latter?

MENSURATION

DEFINITION. *Mensuration* is that branch of mathematics which has for its object the measurement of geometrical magnitudes or quantity.

The science of mathematics is a very old one. It is the oldest of the sciences.

The Pyramids of Egypt are evidence of the age of this science.

The presence of old mathematical manuscripts is further evidence. The oldest manuscript of this kind has the following title: "Directions for knowing all dark things." It is now in the British Museum, and is thought to have been written as early as 1700 B.C., or over 3500 years ago. The author of this work was an Egyptian priest named Ahmes. He treated of Arithmetic, Algebra, Geometry, and Trigonometry. His work treats of such a range of mathematics that one is led to believe that the science of mathematics was well developed before his time.

A third evidence is found in the office of surveyor, an office which is a very old one. An officer called *surveyor* flourished in the region of the Nile River, where civilization existed at a very early period.

The building of great canals, tunnels, bridges, buildings, and other structures that depend upon mathematical knowledge is evidence that the science of mathematics is still flourishing in a high state of development.

By the magnitude of anything is meant its extent. To measure any magnitude is to find how many times it contains a *known* magnitude of the same kind. This known magnitude is called a *unit* of measure. It is necessary that this unit be *known*. The unit of Spanish money is the *peseta*. The statement that the value of a carriage is 500 pesetas is meaningless to any one who does not know the value of a peseta.

It is approximately 117 Km. in distance from Terre Haute to Indianapolis. This is meaningless to any one who does not know the length of a Km.

Units are of two kinds: (1) *Natural* units; as apple, jack-knife, tree. (2) *Artificial* units; as foot, ounce, bushel.

According to the way in which a quantity is measured it may be regarded as one of two kinds: *discrete* or *continuous*.

When a quantity is measured by means of a *natural* unit, *one* of the quantity, it is regarded as *discrete*. Thus, 75 trees, 25 jack-knives, and 34 apples are discrete quantities. In a discrete quantity the unit of measure is separate and distinct.

When a quantity is measured by means of an *artificial* unit, a *portion* of it, the quantity is regarded as *continuous*. Thus, 5 bushels of apples, 14 ounces of tea, and 17 yards of cloth are continuous quantities.

If there are added to a city each year 1200 houses, the quantity is regarded as discrete, being measured by the unit house; while if the value of the 1200 houses is $3,000,000, the quantity is regarded as continuous, being measured by the unit dollar.

Artificial units are usually *standard*, while natural units are not.

DIRECT AND INDIRECT MEASUREMENT

Direct measurement is measurement in which the unit is applied directly to the thing to be measured. The length of a room, the distance between two cities, and the number of yards in a bolt of calico are found by direct measurement.

The distance from the earth to the moon, the height of a mountain, and the cubical contents of a room are all found by *indirect* measurement, which presupposes direct measurement for data and requires some calculation. It is in *indirect measurement* that the science of mathematics has its origin.

Lines, surfaces, solids, angles, value, and time each have magnitude and therefore can be measured.

LINEAR MEASUREMENT

In the measurement of lines the unit is a *line*. All easily accessible straight lines are measured by the direct application of the unit, and the simplest mathematical act — that of counting. For such measurements no rule need be developed. There is very little science involved. The measurement of lines by *indirect* measurement does not belong to ordinary arithmetic.

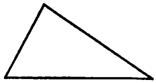
SURFACE MEASUREMENT

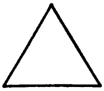
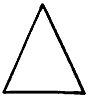
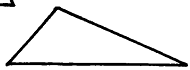
The unit for this is a portion of surface in the form of a square, each side of which is some linear unit. For any particular class of measurements the most convenient square may be chosen. In general, surfaces are measured by indirect measurement.

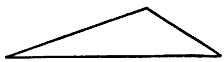
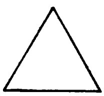
The various forms of plane surfaces are: triangles, quadrilaterals, polygons of more than four sides, and surfaces

bounded by curved lines. The circle is the only form of the last-mentioned that is of interest here.

TRIANGLES

If a plane figure is bounded by three straight lines, it is called a *triangle;* really it is a trilateral; thus, 

If a $\triangle$ has three equal sides, it is an *equilateral* $\triangle$; thus, . If two of its sides are equal, it is an *isosceles* $\triangle$; thus,  If no two sides of it are equal, it is a *scalene* $\triangle$; thus, 

If each of the angles of a $\triangle$ is less than a right angle, the $\triangle$ is called an *acute-angled* $\triangle$; thus, . If one of the angles of a $\triangle$ is a right angle, the $\triangle$ is called a *right-angled* $\triangle$; thus,  If one angle is an obtuse angle, the $\triangle$ is an *obtuse-angled* $\triangle$; thus, . If the angles are all equal, the $\triangle$ is *equiangular;* thus, 

A $\triangle$ can have but *one* right angle, or but *one* obtuse angle. There are no $\triangle$s other than those described above.

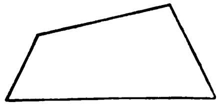
Triangles cannot be measured by direct measurement. Some quadrilaterals can be so measured.

Hence, in the study of the *mensuration* of plane figures, quadrilaterals should be studied before triangles.

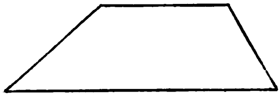
QUADRILATERALS

If a plane figure is bounded by *four* straight lines, it is called a *quadrilateral.*

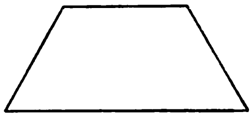
If no two sides of a quadrilateral are *parallel,* it is called a *trapezium;* thus,



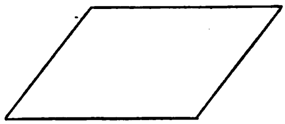
If two sides are parallel, the quadrilateral is called a *trapezoid;* thus,



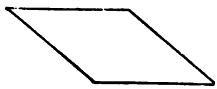
If the two sides of a trapezoid which may not be parallel are equal, it is an *isosceles* trapezoid; thus,



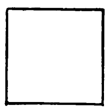
If the two pairs of opposite sides of a quadrilateral are parallel, the figure is called a *parallelogram;* thus,



If the sides of a parallelogram are all equal, it is a *rhombus;* thus,



If the angles of a rhombus are right angles, the figure is a *square;* thus,



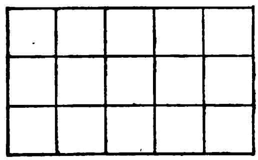
If the angles of a parallelogram are right angles, it is a *rectangle;* thus,

Hence, a square is an equilateral rectangle or an equiangular rhombus.

As is the custom in the mensuration of the quadrilateral, the rectangle will be studied first, because it is the only

plane figure which can be measured by direct measurement; it is the only plane figure that can be exactly measured by applying the unit (a square) directly and counting the number of times it is applied.

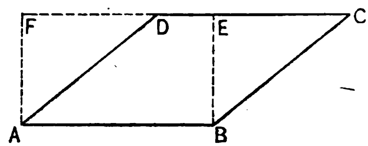
The figure at the right shows how a rectangle may be dissected to show the unit a square. When this is done, counting completes the measurement. The figure contains the unit 15 times; hence, the area is 15 square units. But the rectangle is 5 linear units long and 3 wide. Now, $3 \times 5 = 15$, which number



agrees with the number of square units found by counting. Hence, the rule for finding the area of a rectangle when its length and width are known: Multiply the *number* of units in its length by the *number* of units in its width, and call the result square units. For example, find the area of a rectangle which is 21 ft. long and 10 ft. wide. $21 \times 10 = 210$. $\therefore$ the area is 210 sq. ft. Formula: Area $= ab$ sq. ft. where a and b equal the length and width in feet.

A square is a rectangle, hence the same rule applies. But the length and width are equal; hence, to find the area of a square, square the number of units in its length and call the result square units. For example, the side of a square is 11 in. $\overline{11}^2 = 121$. $\therefore$ the area $= 121$ sq. in. Formula: Area $= a^2$ where a is the length of one side.

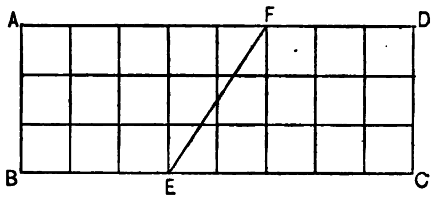
The area of a parallelogram which is not a rectangle cannot be found by direct measurement. *ABCD* is such a parallelogram, and it may be changed to the rectangle *ABEF* by cutting off *BEC* and putting it in as *AFD*. (Use cardboard to show



this.) But the area of *ABEF* equals the length multiplied by the width. Hence, the area of *ABCD* may be found in the same way, for the length, width, and area are all the same in both figures. To one who has studied geometry, the proof that *ABCD* is equal in area to *ABEF* is very easy.

THE MENSURATION OF THE TRAPEZOID

The rectangle *ABCD* contains 24 square units. The portion *ABEF* is equal in area to the portion *ECDF*. (Cut cardboard to show this.) Hence, *ABEF* is equal in area to one-half the area of *ABCD*. Hence, *ABEF* contains 12 square units.



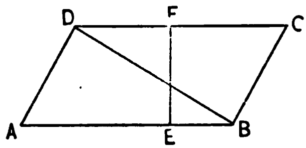
But *ABEF* is a trapezoid by definition, and *AF* plus $BE = BC = AD = 8$ units. By taking the product of one-half the sum of the parallel sides, *AF* and *BE*, of the trapezoid *ABEF*, by the width 3 the result is 12, which is the same number as that obtained above. Hence, the rule for finding the area of a trapezoid when the lengths of the parallel sides and the width are given is illustrated: Multiply the half-sum of the parallel sides by the width, and call the result square units. For example, the parallel sides of a trapezoid are 17 and 13 inches, and the width is 8 inches. $17 + 13 = 30$. $\frac{1}{2}$ of $30 = 15$. $15 \times 8 = 120$. $\therefore$ The area $= 120$ sq. in. Formula: Area $= \frac{1}{2}(a + b)h$, where a and b are the parallel sides, and h the height or width. This formula holds good for the above figure. Others should be tried.

THE TRAPEZIUM

The area of the trapezium can be most easily obtained after the mensuration of the triangle has been studied, since it can be divided into triangles.

THE MENSURATION OF THE TRIANGLE

Draw the diagonal of the parallelogram *ABCD*, thus dividing it into two triangles, *ABD* and *BDC*. By the use of cardboard, or reference to geometry, these triangles can be shown equal to each other. Hence, one of them, as *ABD*, is half of the parallelogram *ABCD*. But the parallelogram *ABCD* equals the product of *AB* and *EF*, the base and altitude.



Since the base and altitude of the △ *ABD* are the same as those of the ▱ *ABCD*, and the △ is one-half of the ▱, it must follow that the area of the △ is *one-half* the product of its base and altitude, which is the rule usually given.

Formula: Area of △ $= \frac{1}{2}ab$.

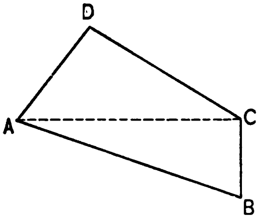
The altitude of a △ is the perpendicular distance from a vertex of the △ to the opposite side.

HERO'S FORMULA

When the lengths of the three sides of a △ are given instead of the base and altitude, use the formula: Area of △ $= \sqrt{s(s-a)(s-b)(s-c)}$, when a, b, and c are the sides, and s their half-sum. This formula is known as Hero's. Hero of Alexandria lived about 110 B.C. The

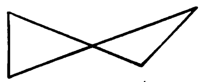
proof which he is supposed to have made for this formula is given in full in W. R. Ball's "History of Mathematics."

THE MENSURATION OF THE TRAPEZIUM

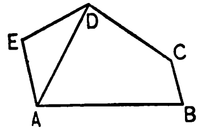


The area of a trapezium may now be found. For example, the trapezium $ABCD$ may be divided into triangles by the line AC, called a diagonal. If the lengths of AB, BC, CD, DA, and AC are known or can be measured, the areas of the two triangles may be found by Hero's formula.

Such quadrilaterals as have been studied so far are called *convex* quadrilaterals.

A plane figure of the form  is called a *concave* quadrilateral, and one of the form  is called a *cross* quadrilateral. The mensuration of such figures will not be attempted here.

POLYGONS

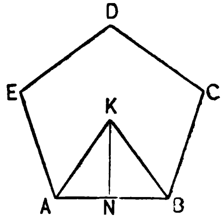
A plane figure of more than four sides is called a *polygon;* thus, 

.

A polygon of five sides is called a *pentagon;* one of six sides, a *hexagon;* of seven sides, a *heptagon;* and one of n sides, an *n-gon.*

The line which joins any two alternate vertices of a polygon, as AD, is called a *diagonal* of the polygon.

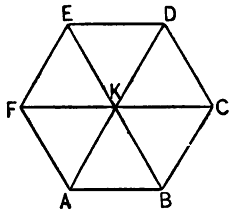
A polygon is *regular* if it is equilateral and equiangular. If K is the center of the regular polygon $ABCDE$, and KN is perpendicular to AB, KN is called an *apothem*. The distance from the center to the sides of a regular polygon are all equal. Hence, the area of *any* regular polygon may be found if the length of one side and that of the apothem are given, because it consists in finding the area of a $\triangle$ when the base and altitude are given, as AB and KN of $\triangle$ ABK.



The length of the apothem of any polygon depends upon the length of the side, and a knowledge of geometry is required to determine the relation.

No polygon except those that are regular need be studied in Arithmetic, and only a few of those.

The regular *hexagon* is an interesting polygon. It is composed of six equilateral triangles, equal each to each. Hence, the area of the hexagon can be found if one side only is known, as AB, since $AB = BK = AK$. Use Hero's formula, or from a knowledge of geometry the apothem may be found. When the area of one $\triangle$ is found, multiply by 6 to find the area of the hexagon.



The line which joins the center of a regular polygon to the vertex of any angle is called the *radius* of the polygon. The area of a regular twelve-sided polygon (dodecagon) $= 3\,r^2$, where r is the radius of the polygon.

THE CIRCLE

The only surface form remaining which should be studied here is the *circle*.

The circle is a plane figure bounded by a curved line such that every point in it is equally distant from a point within called the *center*. The distance from the center to the boundary line is called the *radius*, and the bounding line is called the *circumference*. Any straight line joining two points on the circumference is called a *chord*. A *segment* of a circle is the portion of it bounded by an arc and its chord. A *sector* of a circle is the portion of it bounded by two radii and their intercepted arc.

The ratio of the circumference of a circle to its diameter, $2r$, is represented by the character π, pronounced *pī*, and is equal to 3.1416 nearly. Hence, the circumference of a circle is $2\pi r$; or, since $2r = \text{diameter} = d$, the circumference $= \pi d$. The history of π is very interesting. The ratio of the circumference of a circle to its diameter has engaged the attention of the mathematical world since long before the time of Christ, and no one has yet found the numerical value of it exactly as expressed in the Arabic system. π has been proven incommensurable.

In 1 Kings vii. 23, we read, "And he made a molten sea, *ten* cubits from brim to brim, and a line of *thirty* cubits did encompass round about it." From this Scriptural reference $\pi = 3$.

The Chinese said this ratio was 3.

Ahmes, the author of the oldest mathematical manuscript in existence, gave $\pi = 3.160\dot{4}$. Archimedes over two thousand years ago announced that π was approximately $\frac{22}{7} = 3\frac{1}{7} = 3.1428 \cdots$. Metius about three hundred years ago calculated $\pi = \frac{355}{113}$, which reduced to the decimal

form = 3.14159. Ludolph calculated its value to a great degree of accuracy, and π is therefore often called "The Ludolphian Number."

In later years Richter and Shanks have calculated the value of π to five hundred and seven hundred decimal places respectively.

Since the value of π cannot be found exactly as expressed in Arabic notation, the exact area of a circle cannot be found. Geometry proves that the area of a circle equals πr^2, where r is the radius, *whatever* the value of π may be. If π, which is a factor of the area, is defective in accuracy, πr^2 is not accurate. The area of a circle is sometimes given $\frac{1}{4}\pi d^2$, which in language means "square the diameter and multiply by .7854," for $\frac{1}{4}\pi = \frac{1}{4}$ of $3.1416 = .7854$.

It is also given that the area of a circle $= \frac{1}{2} rc$, where $c =$ circumference. But $c = 2\pi r$, whence $\frac{1}{2} rc = \frac{1}{2} r(2\pi r) = \pi r^2$.

The area of a sector of a circle can be found easily if the radius of the circle and the length of the arc of the sector are known, for the ratio of the whole circumference to the arc is equal to the ratio of the area of the circle to the area of the sector. Why?

EXERCISES

NOTE. Make for each exercise a drawing that shall fairly represent the conditions of the exercise.

1. Find the area of a rectangle 4 ft. 6 in. long and 2 ft. 2 in. wide.

SOLUTION. 4 ft. 6 in. $= 4\frac{1}{2}$ ft. 2 ft. 2 in. $= 2\frac{1}{6}$ ft. $4\frac{1}{2} \times 2\frac{1}{6} = 9\frac{3}{4}$.
$\therefore$ the area $= 9\frac{3}{4}$ sq. ft.

2. How many acres in a field in the form of a parallelogram, if it is 44 rd. long and 35 rd. $5\frac{1}{2}$ ft. wide?

3. How many square feet in a square floor, if it is 14 ft. 5 in. long?

4. Find the area in square feet of a trapezoid if the lengths of the parallel sides are 10 ft. 5 in. and 7 ft. 3 in., and the distance between them is 6 ft. 4 in.

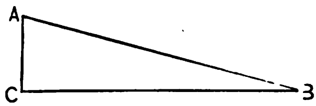
5. Find the area of a triangle if the base is $11\frac{1}{2}$ units long and the altitude is $7\frac{1}{3}$ units long.

6. Find the area of a triangle the lengths of whose sides are 26, 24, and 10 in., respectively.

SOLUTION. Area of $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$. Suppose $a = 26$, $b = 24$, and $c = 10$; then $s = \frac{1}{2}(26 + 24 + 10) = 30$, and $(s - a) = 4$, $(s-b) = 6$, and $(s-c) = 20$. $\therefore \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{30 \times 4 \times 6 \times 20}$ $= \sqrt{2 \cdot 3 \cdot 5 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 5} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$. $\therefore$ the area $= 120$ sq. in.

7. Find the area of a triangle the lengths of whose sides are respectively 34 m., 30 m., and 10 m.

8. If the sides of a triangle are 3, 4, and 5 units long, respectively, it is a *right* triangle, because $3^2 + 4^2 = 5^2$, which general fact was discovered by Pythagoras as early as 600 B.C. From this Pythagorean proposition the length of any side of a right triangle may be found when the lengths of the other two sides are known.



Thus, in the $\triangle$ ABC, with right angle at C, if $AC = 8$, and $BC = 15$, the side AB (called the hypotenuse) $= 17$. For $8^2 + 15^2 = 289$, and the $\sqrt{289} = 17$.

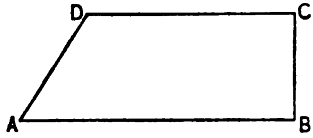
9. Find the length of the hypotenuse of a right triangle if the lengths of the other two sides are 5 ft. and 12 ft.

10. Find the length of the hypotenuse if the other sides are 48 in. and 64 in.

11. Find the length of one side of a right triangle if the hypotenuse and the other side are 20 and 16.

12. The parallel sides of an isosceles trapezoid are 15 ft. and 7 ft. respectively, and the length of each of the other sides is 5 ft. Find its area.

13. If the trapezoid $ABCD$ has right angles at B and C, and $AB = 12$ ft., $CD = 8$ ft., and $AD = 5$ ft., find the area.



14. In the trapezium $ABCD$, if $AB = 10$, $BC = 14$, $CD = 12$, $DA = 8$, and diagonal $BD = 16$ ft., respectively, find the area.

15. Could the diagonal BD in Ex. 14 equal 20 ft.? Why?

16. If the side of a regular hexagon is 6 in., find its area.

17. Find the circumference of a circle whose radius is $10\frac{1}{2}$ in.

Use $\pi = 3.1416$, and $\frac{22}{7}$, and compare results.

18. Find the area of the circle mentioned in (17) in three different ways.

19. If the area of a rectangle is 128 sq. ft. and the length 16 ft., find the width.

The area $= ab$. $\therefore ab = 128$; but $b = 16$. $\therefore 16a = 128$ and $a = \frac{128}{16} = 8$. $\therefore$ the width $= 8$ ft.

20. The area of a parallelogram is $141\frac{3}{4}$ sq. in., and the length of the base is $12\frac{1}{2}$ in. Find the width.

Use the arithmetic equation, as in Ex. 19.

21. The area of a trapezoid is 170 square units, and the lengths of the parallel sides are 21 ft. and 13 ft. Find the width.

22. The area of a trapezoid is 150 sq. ft. and the width is 10 ft. If the lengths of the parallel sides are as $1:2$, find their lengths.

23. The area of a triangle is $14\frac{1}{2}$ sq. ft., and the length of the base is 6 ft. Find the length of the altitude upon the base.

24. Draw the three altitudes of an acute-angled triangle.

25. Do the same for a right and for an obtuse triangle.

26. The area of an equilateral triangle is $25\sqrt{3}$. Find the length of one of its sides.

27. The area of an isosceles trapezoid is 33 sq. ft.; the lengths of the parallel sides are 7 ft. and 15 ft. Find the width and the length of one of the non-parallel sides.

28. The area of an isosceles trapezoid is 40 sq. in.; each of the non-parallel sides is 4 ft. Find the lengths of the parallel sides.

29. The area of a regular hexagon is $384\sqrt{3}$ square units. Find the length of one side.

30. The area of a circle is 113.0976 square units. Find the length of the radius.

31. Find the area of a sector of a circle if its central angle is 30° and the radius is 8 in.

32. If the radius of a circle is 24 in., find the area of a sector of it whose central angle is 50°.

33. If the radius of a circle is 20 in., find the area of a sector whose arc is 40 in. long.

34. A parallelogram has two altitudes. Draw a parallelogram and its two altitudes.

35. The adjacent sides of a parallelogram are 84 ft. and 72 ft. respectively. The area of it is 3360 sq. ft. Find the lengths of its two altitudes.

36. Find the area of a square of equal perimeter to that of the parallelogram mentioned in Ex. 35.

37. The area of a rectangular field is 20 A., and it is two times as long as wide. Find the length and width in rods.

38. The area of a rectangular field is $12\frac{1}{2}$ A., and its width is $\frac{4}{5}$ of its length. Find the dimensions of the field in rods.

39. The area of a rectangular field is 360 A., and the length is to the width as 16 : 9. Find the dimensions of the field.

40. In a rectangular board 6 ft. long and 15 in. wide there are two round holes, one of which is 14 in. across and the other 10 in. across. Find the area remaining.

41. Find the area of the circular ring between two concentric circles whose circumferences are 154 in. and 210 in. respectively.

42. Find the area and the circumference of a circle inscribed in a square containing $156\frac{1}{4}$ sq. m.

43. Find the area of a circle circumscribed about a square containing 110.25 sq. cm.

44. Find the area inclosed by three equal circles which touch each other externally, the radius of each being $3\frac{1}{2}$ ft.

SIMILAR SURFACES

A given rectangle is 5 ft. long and 2 ft. wide. Another rectangle is 10 ft. long and 4 ft. wide. The ratio of the lengths of these two rectangles equals the ratio of their widths. Thus, 5 ft. : 10 ft. = 2 ft. : 4 ft.

When the lengths of two rectangles have the same ratio as that of their widths, the rectangles are said to be *similar*.

The rectangle which is 5 ft. by 3 ft. is *similar* to one which is 15 ft. by 9 ft.; as also one which is $12\frac{1}{2}$ in. by $7\frac{1}{2}$ in., or one 25 m. by 15 m. Why?

A triangle whose three sides are 4 in., 3 in., and 2 in. long respectively is *similar* to one whose sides are 8 in., 6 in., and 4 in. long. It is similar to one whose sides are 10 in., $7\frac{1}{2}$ in., and 5 in. long.

One triangle is *similar* to another when the corresponding sides are proportional, or they are similar when the ratios of their corresponding sides are equal.

If the sides of one triangle are 7 in., 5 in., and 8 in., respectively, and the sides of another are 21 in., 15 in., and 24 in., they are similar, for 7 in. : 21 in. = 5 in. : 15 in. = 8 in. : 24 in.

Any two polygons are similar if their corresponding sides are proportional and their corresponding angles are equal. Any two squares are similar, and any two regular pentagons are similar. Why?

The area of a rectangle 4 ft. by 3 ft. is 12 sq. ft. The area of a rectangle which is 8 ft. by 6 ft. is 48 sq. ft.

These two rectangles are similar, and the ratio of the dimensions of the second one to the corresponding dimensions of the first is 2. But the area of the second is 4 times that of the first.

The ratio of their areas is the *square* of the ratio of their corresponding dimensions. (The dimensions of a plane figure are its length and width.)

The area of a rectangle 3 ft. by 2 ft. is 6 sq. ft. The area of one 12 ft. by 8 ft. is 96 sq. ft. The ratio of the dimensions of the second to those of the first is 4, while the ratio of the area of the second to that of the first is 16. Again, the ratio of their areas is the *square* of the ratio of their corresponding dimensions.

By Hero's formula, the area of a triangle whose sides are 13, 14, and 15 units long is 84 square units; by the same formula, the area of a triangle whose sides are 26, 28, and 30 units long is 336 square units.

The ratio of their areas is the *square* of the ratio of their corresponding dimensions.

The following general law regarding similar surfaces is true for all of those discussed above: The areas of any two similar surfaces are to each other as the squares of any two corresponding dimensions.

EXERCISES

1. Find the area of a rectangle 5 ft. by 4 ft.

2. Multiply the dimensions of the rectangle in Ex. 1 by 2; by 3; and by 4. What are the areas of the resulting rectangles? How were these results obtained?

3. Multiply the dimensions of the rectangle in Ex. 1 by $2\frac{1}{2}$; by $\frac{2}{3}$; and by $\frac{3}{7}$. What are the areas of the resulting rectangles?

4. The area of a triangle whose sides are 3 cm., 4 cm., and 5 cm. long is 6 sq. cm. Multiply the lengths of the sides of this triangle by 3, and find the area of the resulting triangle.

5. Give two ways for finding the area of the resulting triangle in Ex. 4.

6. Find the area of a triangle whose sides are 17, 17, and 16 units long.

7. If another triangle has its sides 14 times the lengths of the three sides of the triangle in Ex. 6, what is the area of it?

8. If the sides of a triangle are respectively n times the lengths of the sides of another triangle, compare their areas.

9. If the lengths of the sides of a triangle are 8, 15, and 20, and the side of another triangle corresponding to the side 8 in this one is 12, find the lengths of the other two sides if the triangles are similar.

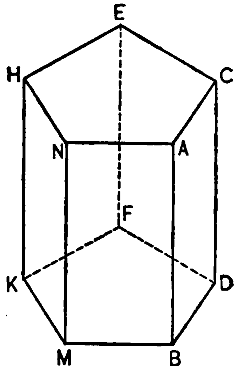
MENSURATION OF SOLIDS

The unit in the measurement of solids is itself a solid in the form of a cube, each edge of which is some linear unit.

A cube is a solid bounded by six equal faces, each in the form of a square.

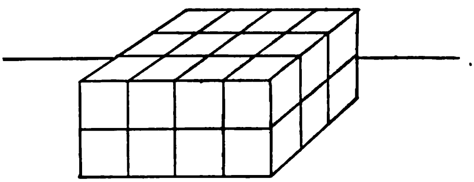
An edge of a cube is one side of one of the square faces. A cube has twelve edges.

The first solid studied should be such as can be dissected into cubes, that the simple act of counting may be used in measuring it. The rectangular prism is such a solid.



A prism is a solid which has equal and parallel polygons for its bases, and whose faces are parallelograms. A prism may have any kind of a polygon for a base.

Hence, there is as great a variety of prisms as there is of forms of polygons. The illustration above is a pentagonal prism, because its base is a pentagon. *AC*, or *BD*, is a *base* edge, and *AB* is a *lateral* edge.

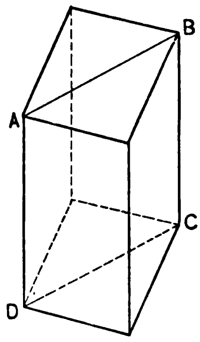


A rectangular prism has all of its faces and bases in the form of a rectangle; hence, it can be dissected to show the measuring unit, the cube, as shown above.

By counting, it is found to contain 24 cubic units. If the *number* of square units in the base is multiplied by the *number* of linear units in the height, the resulting number is the same as the number of cubic units found by counting.

Hence, to find the volume of this prism, multiply the number of square units in the base by the number of linear units in the height, and call the result cubic units.

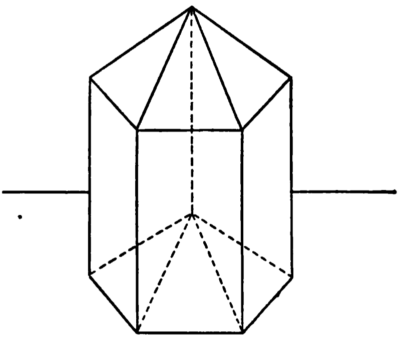
The figure at the right shows how a rectangular prism may be dissected to show two equal triangular prisms, by passing a plane through it diagonally. *ABCD* is such a plane.



Each triangular prism is one-half of the given prism, and the base of the triangular prism is one-half of the base of the given prism. The altitude of each is the same.

Hence, to find the volume of the triangular prism, use the same rule as for the rectangular prism.

The figure at the right shows how a prism with any polygon for a base may be divided into triangular prisms having the same altitude as the given prism, and the sum of whose bases equals the base of the given prism.



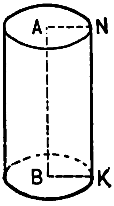
Hence, to find the volume of *any* prism, multiply the number of square units in the base by the number of linear units in the height, or altitude, and call the result cubic units.

The lateral surface of a prism is made up of the lateral faces, which are always parallelograms. The bases of the parallelograms make up the perimeter, or bounding line of the base of the prism.

Hence, the usual rule for finding the lateral area.

State a rule for finding the total area of a prism.

If a solid has two parallel and equal circles for bases, and the diameter is uniform throughout, the solid is called a *circular cylinder*.

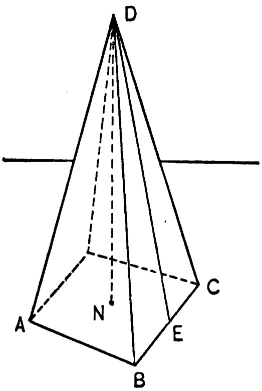


A *right* circular cylinder is one in which the axis *AB* is perpendicular to the plane of the base.

A cylinder of revolution is one formed by turning a rectangle, as *ABKN*, around *AB* as an axis. It becomes a *right circular cylinder*. There are other cylindrical forms, but they will not be treated here.

Regarding a cylinder as a prism whose base is a polygon with an infinite number of sides, the volume is found by the same rule as that of a prism. State the rule. Give a good explanation of how to find the lateral and entire surfaces of a cylinder. The lateral surface of a prism or of a cylinder is sometimes called the *convex* surface.

PYRAMIDS



A *pyramid* is a solid which has some kind of a polygon for a base and whose sides are triangles which meet in a common point. If the base is a △, the solid is a *triangular* pyramid. If the base is a pentagon, the solid is a *pentagonal* pyramid. The common point at which the sides meet is called the *apex*. The triangular sides are called the *lateral* faces. The boundary lines of the base are called the *base edges* of the pyramid; thus, *AB*. The lines which connect the

apex with the intersection points of the base edges are called the *lateral* edges; thus, *DB*. The straight line which may be drawn from the apex perpendicular to a base edge is called the *slant height* of the pyramid; thus, *DE*.

A *right* pyramid is one in which a line from the apex to the *center* of the base is perpendicular to the plane of the base. The *altitude* of a pyramid is the perpendicular distance from the apex to the plane of the base; thus, *DN*.

A *cone* is a solid which has a circular base and which tapers uniformly to a point called the apex of the cone.

The curved line which bounds the base is called the perimeter of the base.

A *right* cone is one in which a line from the apex to the center of the base is perpendicular to the plane of the base.

If the base of a right cone is a circle, it is the same as a *cone of revolution*, which is made by the revolution of a right triangle around one of its legs as an axis.

Neither the pyramid nor the cone can be measured by the direct application of a cube to it; nor can either be dissected so as to show the unit (a cube).

To measure a pyramid compare it to a prism having the same sized base and an altitude of the same length as that of the pyramid. Take a tin or copper vessel in the shape of a prism with a given base and a given altitude; also a vessel in the shape of a pyramid with base and altitude respectively equal to those of the prism. Then show that the prism-shaped vessel holds just three times as much water as the pyramid-shaped one, and that the usual rule for finding the volume of a pyramid is true. For a more rigorous proof of the rule see any good work on solid geometry.

To find the volume of a cone prepare for the same kind of an experiment as was performed to find the volume of a pyramid, the "pouring experiment." Or see a work on solid geometry for a demonstration.

EXERCISES

Make a good drawing for each exercise.

1. The base of a rectangular prism is 15 ft. by 8 ft., and the altitude of the prism is 12 ft.; find the volume.

SOLUTION. $15 \times 8 = 120$. $120 \times 12 = 1440$. Or, $15 \times 8 \times 12 = 1440$. $\therefore$ the volume = 1440 cu. ft.

2. A right prism has a base in the form of a parallelogram whose length and width are 14 ft. and 8 ft. respectively; if the altitude of the prism is 20 ft., find its volume.

3. The lateral edge of a right prism is 15 ft., and the base is a regular hexagon with one side 8 ft. long; find its volume.

4. Find the volume of a right prism with a triangular base, if the base edges are 7 in., 8 in., and 9 in. long, and the lateral edge is 15 in. long.

5. The radius of a circular cylinder is $38\frac{1}{2}$ in., and the length of the cylinder is 27 ft.; find its cubical contents.

6. A right pyramid has a square base, each side 4 ft. 6 in. long; the altitude of the pyramid is $7\frac{1}{2}$ ft.; find its volume.

7. The altitude of a right prism is 25 in., and the base is a regular hexagon, each side 15 in. long; find its volume.

8. The altitude of a right circular cone is 13 in., and the radius of the base 7 in.; find its volume.

9. The altitude of a right prism with a square base is 8 in., and the volume is 242 cu. in.; find the length of one base.

SOLUTION. $242 \div 8 = 30.25$. $\therefore$ the area of the base is 30.25 sq. in. $\sqrt{30.25} = 5.5$. $\therefore$ the length of one base edge = 5.5 in.

10. The base of a right prism is an equilateral triangle, and the altitude is 15 in. long. If the volume is $135\sqrt{3}$ cu. in., find a base edge.

11. The volume of a right prism with a square base is 432 cu. in. If the length of a lateral edge is 2 times the length of a base edge, find the dimensions of the prism.

12. If a right prism has a square base, and a base edge is 2 times as long as the lateral edge, find the dimensions, if the volume equals 864 cu. m.

13. The volume of a hexagonal right prism is $810\sqrt{3}$ cu. ft., and the lateral edge is 15 ft. Find one of the base edges.

14. The volume of a right pyramid with a square base is 211.25 cu. in., and the altitude is 15 in. Find the length of a base edge.

15. The volume of a right pyramid with a square base is $\frac{27}{2}\sqrt{7}$ cu. in., and the slant height is 4 times as long as a base edge. Find the length of a base edge.

16. The volume of a right circular cone is 11979 cu. m., and the altitude is 42 m. Find the radius of the base. ($\pi = \frac{22}{7}$.)

17. The lateral area of a right circular cone is 220 sq. dm., and the radius of the base is 5 dm. Find the volume in cubic decimeters.

18. The lateral edge of a pyramid with a square base is 15 ft., and each base edge is 9 ft. Find the volume.

19. Find the lateral area of the prism in Ex. 1 of this list.

20. Show that in general the lateral area of the prism in Ex. 2 cannot be determined.

21. Find the lateral area of the prism in Ex. 3.

22. Find the entire area of the surface of the prism in Ex. 4.

23. The altitude of a square pyramid is 12 in., and the base edge is 10 in. Find the lateral surface area.

24. Find the surface of a right circular cone, if its altitude is 15 in. and the radius of its base is 8 in.

25. If the area of the base of a right circular cone is 346.5 sq. dm., and the altitude is 14 dm., find the lateral area.

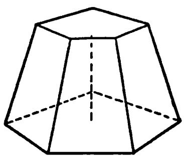
26. The entire surface area of a cube is 1176 sq. in. Find the volume of the cube.

27. Find the altitude and diameter of the base of a cylinder of revolution whose lateral area is 168π, and whose volume is 504π.

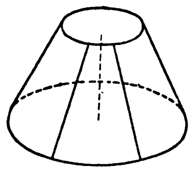
28. Find the volume of a cone of revolution whose slant height is 29 in., and the lateral area is 580π sq. in.

FRUSTUMS OF PYRAMIDS AND OF CONES

If a plane is passed between the vertex and the base of a pyramid or of a cone, parallel to the base, the portion between the base and this plane is called the *frustum* of the pyramid or of the cone.



I



II

Figure I is a frustum of a pyramid; and Figure II is a frustum of a cone. The upper and the lower bases of each frustum are similar figures. The lateral faces of the frustum of a pyramid are trapezoids.

The volume of a frustum $= \frac{H}{3}(B + b + \sqrt{Bb})$, where H is the altitude, B is the area of one base, and b the area of the other base. These bases may be squares, triangles or polygons of any form, or circular. In any instance B and b are similar. In the application of this formula to the

frustum of a cone, the volume $= \frac{H}{3}(\pi R^2 + \pi r^2 + \pi Rr)$, where R and r are the radii of the two bases. $\frac{H}{3}(\pi R^2 + \pi r^2 + \pi Rr) = \frac{\pi H}{3}(R^2 + r^2 + Rr) = \frac{\pi H}{3}\frac{(R^3 - r^3)}{(R - r)}$.

EXERCISES

1. Find the volume of the frustum of a square pyramid, if the side of the lower base is 7 ft., the side of the upper base is 5 ft., and the altitude is 12 ft.

2. Find the lateral area of the above frustum. (Find the slant height first.)

3. Find the volume of the frustum of a cone if the radius of the lower base is 15 in., the radius of the upper base is 9 in., and the height is 18 in. ($\pi = \frac{22}{7}$.)

SOLUTION. Volume $= \frac{H}{3}(B + b + \sqrt{Bb}) = \frac{18}{3}(\frac{22}{7} \times 15 \times 15 + \frac{22}{7} \times 9 \times 9 + \frac{22}{7} \times 15 \times 9) = \frac{22}{7} \times \frac{18}{3}\frac{(\overline{15}^3 - 9^3)}{(15 - 9)} = \frac{22}{7} \times 6 \times \frac{2646}{6} = 8316$.
$\therefore$ vol. $= 8316$ cu. in.

4. Find the lateral area of the frustum mentioned in Ex. 3.

5. The lateral edge of the frustum of a square pyramid is 10 ft., and the sides of its bases are 6 ft. and 3 ft. respectively; find its lateral area.

6. Find the volume of the frustum in Ex. 5.

7. The volume of the frustum of a cone of revolution is 6020π; its altitude is 60; and the radius of the lower base is 15. Find the radius of the upper base.

8. Find the lateral area of the frustum in Ex. 7.

THE SPHERE

The *sphere* is a solid bounded by a surface all points of which are equally distant from a point within called the center.

The volume of a sphere $= \frac{1}{6}\pi d^3$, where d is the diameter of the sphere; or, it is $\frac{4}{3}\pi r^3$, where r is the radius. Neither of these formulas can be proven except by the principles of solid geometry, but they are easily used in practice.

The surface of a sphere $= 4\pi r^2$, or four times the area of a great circle of the sphere.

EXERCISES

1. Find the volume of a sphere whose radius is 14 in.

SOLUTION: Volume $= \frac{1}{6}\pi d^3 = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times 14 \times 14 \times 14 = 11498\frac{2}{3}$ cu. in.

2. Find the surface of the above sphere.

3. How many cubic centimeters are there in a cannon ball whose diameter is 15 cm.?

4. How many square inches of leather will cover a ball $10\frac{1}{2}$ in. in circumference?

5. How many cubic decimeters in a sphere which is the largest one that can be placed in a cubical box 7 dm. in depth?

6. Find the volume of a sphere whose circumference is 31.416 ft.

BOARD MEASURE

The unit in *board measure* is the *board foot*, which is a board 1 ft. square and 1 in. thick. Thus, a board 7 ft. long, 1 ft. wide, and 1 in. thick contains 7 *board feet*.

A board 7 ft. long, 1 ft. wide, and $1\frac{1}{2}$ in. thick contains $10\frac{1}{2}$ board feet. A board 7 ft. long, 9 in. wide, and 1 in. thick contains $5\frac{1}{4}$ board feet. Lumber is usually sold at a certain price per hundred (C), or per thousand (M) board feet (bd. ft.).

EXERCISES

1. How many *board feet* in 32 planks 24 ft. long, 10 in. wide, and 1 in. thick?

SOLUTION. $24 \times \frac{10}{12} = 20$. $\therefore$ each plank contains 20 board feet; and 32 will contain 640 board feet.

2. If joists are 18 ft. 6 in. long, 8 in. wide, and $1\frac{3}{4}$ in. thick, what will 14 such joists cost at $1.90 per hundred board feet?

SOLUTION. $18\frac{1}{2} \times \frac{8}{12} \times 1\frac{3}{4} = 21\frac{7}{12}$. $\therefore$ each joist contains $21\frac{7}{12}$ bd. ft. $21\frac{7}{12} \times 14 = 302\frac{1}{6}$. $\therefore$ $302\frac{1}{6}$ bd. ft.; or $3.02\frac{1}{6}$ C bd. ft., which at $1.90 per C will cost $5.74 to the nearest cent.

3. Find the cost of 29 studding 18 ft. long, 4 in. wide, and 2 in. thick at $1.80 per C.

4. What is the cost of 27 sills 17 ft. 2 in. long, 8 in. wide, and 5 in. thick at $2.10 per C?

5. Boards are in the shape of a trapezoid 14 ft. 6 in. long, 10 in. wide at one end and 7 in. at the other, and $1\frac{1}{4}$ in. thick. How many board feet in 16 such boards?

CARPETING ROOMS

Unless otherwise told, carpet dealers lay carpets lengthwise the room. It is more artistic to do so.

As many strips are required as the width of the room contains the width of a strip. If the width of the room does not contain the width of a strip an integral number of times, the integral number next after the fractional quotient is taken as the number of strips.

If it is required to lay the carpet so as to get the most economical cost, it may be necessary to lay the strips *crosswise* the room.

It is not necessary to find the surface area of a floor, since carpets are not bought by the square yard.

EXERCISES

1. A room is 16 ft. 7 in. long and 14 ft. wide. What will a carpet cost at 92¢ per yard, if the carpeting is 2 ft. 6 in. wide?

SOLUTION. 14 ft. = 168 in. 2 ft. 6 in. = 30 in. $168 \div 30 = 5\frac{3}{5}$. $\therefore$ *six* strips are required. Each strip is 16 ft. 7 in. long = $5\frac{19}{36}$ yards. Then 6 strips will contain $5\frac{19}{36}$ yd. $\times 6 = 33\frac{1}{6}$ yd. $33\frac{1}{6}$ yd. at 92¢ per yd. will cost $30.51.

In general, allowance for matching the carpet strips cannot be provided for in the preparation of problems, so that the actual cost is more than the cost obtained by the above process.

2. How many yards of carpeting 2 ft. 8 in. wide must be purchased to cover a floor 17 ft. 2 in. long and 15 ft. 4 in. wide?

3. A floor is 14 ft. 10 in. long and 13 ft. 5 in. wide. If carpeting is $\frac{5}{6}$ yd. wide, what will it cost to carpet the floor at 87¢ per yard, if laid lengthwise?

4. Find the cost to carpet the floor in Ex. 3 if the strips are laid crosswise, and account for the difference, if any.

5. Suppose there is a loss of $\frac{1}{5}$ of each strip for matching the pattern, find the cost of carpeting a room 18 ft. 3 in. long and 15 ft. 8 in. wide with carpeting 2 ft. 4 in. wide at $1.10 per yard.

6. How many meters of carpet 8 dm. wide does it require for a floor 6 m. long and 480 cm. wide?

7. How much will it cost to carpet a room 54.6 dm. long and 4.7 m. wide with carpet 75 cm. wide at $1.08 per meter?

8. At $2.80 per centare, find the cost of covering a floor which is 6 meters long and 48 dm. wide with carpet 8.5 dm. wide.

9. If it takes 42 m. 25 cm. of carpet to cover a floor 6 m. 2 dm. 5 cm. long and 4.5 m. wide, find the width of the carpet.

PAPERING ROOMS

Wall paper varies in width from 18 in. to 20 in. and even 30 in. But 18 in. should be considered the width unless it is otherwise designated.

A roll of paper contains 8 yd., or 24 ft. Each roll contains 36 sq. ft. and 3 rolls contain 108 sq. ft. But paper dealers say that it is customary to allow three rolls to each *square*, or 100 sq. ft., in practice.

EXERCISES

1. How many rolls of paper are required to paper a room 7 yd. long, $5\frac{1}{2}$ yd. wide, and 4 yd. high?

SOLUTION. The four walls and ceiling are usually papered. The four walls equal a rectangle 25 yd. long and 4 yd. wide, and contain 100 sq. yd. The ceiling contains 38^1 sq. yd. Hence, both contain $138\frac{1}{2}$ sq. yd. Each roll contains 4 sq. yd. $138\frac{1}{2} \div 4 = 34\frac{5}{8}$. Since fractions of rolls cannot be purchased, it will require 35 rolls. Allowing 3 rolls to each 100 sq. ft. it will take at least 37 rolls.

2. What will it cost to paper the above room at 40 cents per roll put on, and put on a border at 20 cents per yard?

3. What will be the cost of papering a room 23 ft. 8 in. long, 20 ft. 6 in. wide, and 9 ft. high at 72 cents per roll put on, deducting one door 7 ft. 6 in. by 2 ft. 3 in. and 3 windows each 5 ft. 8 in. by 3 ft. 3 in.?

4. A room is 11.2 meters long, 96 dm. wide, and 500 cm. high. If a roll of paper is 7 m. long and 5 dm. wide and costs 96 cents per roll put on, find the cost of papering the room after allowing for 2 doors 3.5 m. long and 1.5 m. wide, 3 windows 25 dm. long and 14 dm. wide, and a baseboard 30 cm. high, if a border is put on at 15 cents per meter.

5. Find the cost of papering a hemispherical dome whose diameter is 42 ft. at 44¢ per square yard.

THE LAYING OUT OF PUBLIC LANDS

The **public lands** of the United States of America are generally laid out in squares, the sides of which run approximately north and south, and east and west.

The present system of survey of public lands is due in no small degree to the efforts of Thomas Jefferson, who was appointed chairman of a committee by the Continental Congress in 1784.

The report of this committee, with some amendments, was finally passed as an *ordinance* on the 20th of May, 1785.

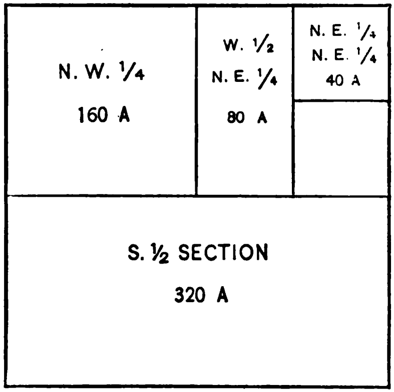
The ordinance provided for townships 6 miles square, containing 36 sections, each 1 mile square. It was further provided that "sections shall be numbered respectively, beginning with the number 1 in the northeast section, and proceeding west and east alternately through the townships, with progressive numbers, till the 36th be completed." This method of numbering the sections of a township is still in use. The accompanying diagram shows the method of numbering the sections.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

An act of Congress in 1805 provided for subdividing the sections into quarter-sections. In 1820 an act provided for the subdivision into *half*-quarter-sections by a

line running north and south, and in 1832 Congress provided that subdivisions should show *quarter*-quarter-sections, and that the line doing this should run east and west.

This diagram shows the divisions and subdivisions of a section, say Section 32, into half-sections, quarter-sections, half-quarter-sections, and quarter-quarter-sections.



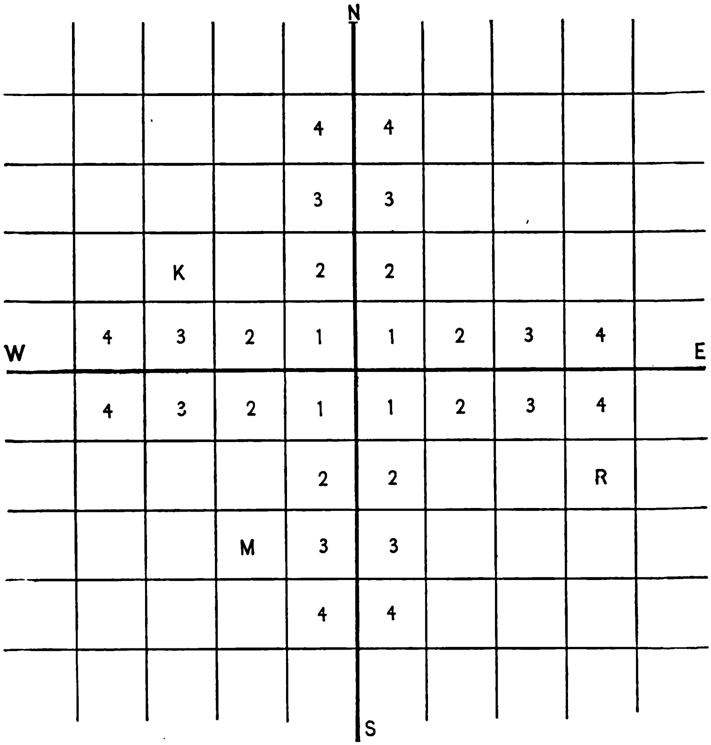
In the upper right-hand corner is found the N.E. $\frac{1}{4}$ of the N.E. $\frac{1}{4}$ of Section 32, containing 40 acres.

The townships are located with reference to a Base Line located on a parallel of latitude, and a Principal Meridian, which lines are established. Then lines are run parallel to these intersecting lines 6 miles apart each way. On page 167, *EW* is a *base line* and *NS* is a *principal meridian*. A line of townships running north and south is called a *range* of townships.

Ranges are designated by their number east or west of the principal meridian. A township is located by giving its *number* north or south of the *base line* and its *range* east or west of the *principal meridian*. Thus, in the diagram township *K* is township 2 north, and range 3 west of P.M. *M* is township 3 south and range 2 west. *R* is township 2 south, and range 4 east. The following is the description found on the face of a deed: "The N.W. $\frac{1}{4}$ of Section 29, township 19 north, range 18 west of the prin-

cipal meridian." This piece of land is found in the western part of the state of Kansas.

Since the earth's surface is spherical, the meridians converge as they approach the poles, and are not truly parallel.



As the bounding lines on the east and west sides of a township are meridian lines, the townships are not perfect squares. This condition necessitates what are known as "corrected lines." This should explain in part why a deed usually says "containing 160 acres, more or less."

EXERCISES

1. Make a diagram showing the location of the following described land: The S.W. $\frac{1}{4}$ and the E. $\frac{1}{2}$ of the N.W. $\frac{1}{4}$ of Sec. 3, township 2 north, range 3 west of P.M.; tell how many acres are in this tract.

2. Show the location of the following farm: The N.W $\frac{1}{4}$ of the S.E. $\frac{1}{4}$ and the N.E. $\frac{1}{4}$ of the S.W. $\frac{1}{4}$ of Sec. 19, township 4 south, range 5 west. How many acres in the farm?

3. Locate the following farm: The E. $\frac{1}{2}$ and the N.W. $\frac{1}{4}$ of the S.E. $\frac{1}{4}$ of Sec. 1, township 1 north, range 18 west. How many acres in the farm?

4. A man owns the N.W. $\frac{1}{4}$ of the S.E. $\frac{1}{4}$, the S.W. $\frac{1}{4}$ of the N.E. $\frac{1}{4}$, the S.E. $\frac{1}{4}$ of the N.W. $\frac{1}{4}$, and the N.E. $\frac{1}{4}$ of the S.W. $\frac{1}{4}$ of Sec. 13, township 2 south, range 1 east. Locate the entire farm, and tell how many acres in it.

5. Mr. Jones owns the S.W. $\frac{1}{4}$ of the S.W. $\frac{1}{4}$ of Sec. 13, and the N.W. $\frac{1}{4}$ of the N.W. $\frac{1}{4}$ of Sec. 24, all in township 3 north, range 4 west. Do these two pieces of land join each other? Locate the entire farm, and tell the number of acres in it.

6. A landowner possesses the following: The S.W. $\frac{1}{4}$ of Sec. 16, and the N.E. $\frac{1}{4}$ of the N.E. $\frac{1}{4}$ of Sec. 20; also the S.W. $\frac{1}{4}$ of the N.E. $\frac{1}{4}$ of Sec. 20. Can the owner go from any one of these three pieces of land to another, without going off his land? Make a diagram to support your decision. How many acres does he own in all?

SPANISH LAND MEASURES

A brief reference to Spanish Land Measure is here made on account of our new possessions, as a result of the war with Spain.

The *unit* is the *vara*. It is a linear unit and is about 33 inches long, though its length varies in different localities.

The unit of area is the *labor*, which is a square, each side of which is 1000 varas long. A *square league* is a surface unit each side of which is 5000 varas in length. Hence, there are three units of area: the *square vara*, the *labor*, and the *square league*. One acre in the common system = 5645.376 square varas in the Spanish system.

SPECIFIC GRAVITY

Weigh a piece of iron very accurately. Then place the piece of iron in a jar *full* of water. Some water will run out — an amount equal to the bulk of iron. Now weigh very accurately the water that ran out. Divide the weight of the piece of iron by the weight of the water displaced by the iron. The quotient expresses the number of times the weight of the iron is of the weight of an *equal bulk* of water. The quotient is called the *specific gravity* (sp. gr.) of iron, for water is taken as the standard of reference. The specific gravity of water is taken as 1.

The sp. gr. of iron is about 7.21; that is, a given bulk of iron weighs 7.21 times as much as an equal bulk of water.

Platinum is the heaviest known substance and has a sp. gr. of 23.

The sp. gr. of gold is 18.5	The sp. gr. of boxwood is 1.28
The sp. gr. of mercury is 13.6	The sp. gr. of milk is 1.03
The sp. gr. of lead is 11.4	The sp. gr. of satinwood is .96
The sp. gr. of silver is 10.5	The sp. gr. of ice is .92
The sp. gr. of copper is 8.8	The sp. gr. of oil is .9
The sp. gr. of zinc is 7	The sp. gr. of alcohol is .8
The sp. gr. of granite is 2.6	The sp. gr. of ash is .6
The sp. gr. of slate is 2.5	The sp. gr. of cork is .24

Any substance whose sp. gr. is greater than 1 would sink if placed in water, and one whose sp. gr. is less than 1

would float. The sp. gr. of any substance in the preceding table may vary from what is given there, but the results are supposed to be fairly accurate.

In the common system, 1 cu. ft. of water weighs 1000 oz. avoirdupois = $62\frac{1}{2}$ lb.; in the metric system, 1 cu. cm. of water weighs 1 g.; 1 l. weighs 1 Kg.; 1 cu. m. weighs 1 metric ton.

In taking the sp. gr. of the gases, *hydrogen* is taken as the standard of reference.

EXERCISES

1. What is the weight of a cubic foot of granite?

2. What is the weight of a cubic foot of lead?

3. What is the weight of a cubic decimeter of zinc?

4. What is the weight of 1 l. of alcohol?

5. Of a cubic foot of silver? Of a cubic foot of ice? Of 1 l. of mercury? Of 1 gal. of milk? Of 1 cu. ft. of cork?

6. What is the weight of a bar of iron 5 in. long, 4 in. wide, and $2\frac{1}{2}$ in. thick?

7. 1 cu. ft. of stone weighs 185 lb. Find its sp. gr.

8. Find the sp. gr. of a liquid weighing 10 lb. per gallon.

9. Find the weight of 6.32 Hl. of olive oil (sp. gr. = .915).

10. If 27 l. of alcohol weigh 22.14 Kg., find the sp. gr.

11. If a stone weighs 8.42 Kg. in air, and 5.32 Kg. in water, find its sp. gr.

12. A plate of metal 12.37 m. long, 8.15 m. wide, and 18.35 dm. thick, weighs 475 Kg. Find its sp. gr.

13. Find the volume of 340 lb. steel (sp. gr. = 7.8).

14. Find the volume of 5 metric tons gunpowder, if its sp. gr. is .93.

15. How many Dl. of alcohol will weigh 800 Kg.?

16. Find the weight of 6400 cu. cm. of cork.

17. Find the volume of 750 Kg. of indigo (sp. gr. = .77).

18. Find the sp. gr. of naphtha, if 10 cu. cm. weigh 85 dg.

19. If 20 cu. ft. of ebony weigh 1487 lb. 8 oz., find its sp. gr.

20. A block of marble containing 7 cu. ft. 860 cu. in. weighs 1265 lb. 10 oz. Find its sp. gr.

MISCELLANEOUS EXERCISES IN MENSURATION

1. A rectangular garden is 40 ft. long and 24 ft. wide. It is surrounded by a walk 4 ft. wide. What will the walk cost at 81¢ per square yard?

NOTE. It is advisable to make a careful drawing in all exercises in mensuration.

2. A map is $2\frac{1}{4}$ ft. by $1\frac{1}{2}$ ft. If the scale of the map is 2 mi. to the inch, how many square miles of country does the map represent?

3. A patch of ground is in the form of a trapezoid, and contains 2 A. 120 sq. rd. If one of the parallel sides is 270 yd. long, and the distance between them is 150 ft., find the length of the other parallel side in rods.

4. How many paving stones, each 8 in. by 4 in., are required to pave $1\frac{1}{2}$ mi. of street 70 ft. wide?

5. In the right triangle ABC, right angle at C, if $AC = 5$, and $BC = 12$, find the length of AB. Who was the discoverer of the proposition which must here be used?

6. If $AB = 7\frac{1}{2}$, and $BC = 6$, find AC. (See Ex. 5.)

7. How far from a tower 35 ft. high must the foot of a ladder 45 ft. long be placed just to reach the top of the tower?

8. A pole was broken 26 ft. from the foot of it, and fell so that the end struck 19 ft. 6 in. from the foot. Find the length of the pole.

9. A walk 5 ft. wide surrounds a circular pond 75 ft. in diameter. Find the number of square feet in the walk.

10. The diameter of a circle is 42 in. Find the area and the perimeter of a square equal in area to it.

11. The side of a square is 21 ft. 6 in. Find the radius of a circle equal in area to it.

SOLUTION. The area of the square $= 21\frac{1}{2} \cdot 21\frac{1}{2}$ sq. ft. $= 462.25$ sq. ft. But $\pi r^2 =$ the area of a circle. $\therefore \pi r^2 = 462.25$. $\therefore r^2 = 462.25 \times \frac{7}{22} = 147.08$, to the nearest hundredth. $\therefore r = 12.12$ ft.

12. Find the diagonal of a room 15 ft. long, 15 ft. wide, and 9 ft. high.

13. The diagonal of a cube $= 10$ cm. Find the length of an edge to the nearest thousandth.

14. The radii of two concentric circles are 6 m., and 4 m. 5 dm. long, respectively. Find the area of the ring, in square meters, formed by these circles.

15. A horse is tied to a stake in the corner of a rectangular field by a rope 45 ft. long. Over how many square feet can the horse graze?

16. Suppose the rope in Ex. 15 is tied to the top of a stake 27 ft. high. Over how many square feet can he graze?

17. A man swims at right angles to the bank of a river at the rate of 10.5 mi. in 3 hr. in still water. The rate of the current is 9 mi. per hour. Find the rate of the swimmer's motion.

18. If the river in Ex. 17 is 1000 ft. wide, how far downstream will the swimmer be when he reaches shore?

19. The roof of a barn is "comb-shaped," and is 60 ft. long; the rafters are 18 ft. long; it is covered with shingles which average 4 in. in width and are exposed 5 in. to the weather. If two rows are laid at the lower edges of the roof, how much will the shingles cost at $ 2.50 per M?

20. A field is 80 rd. long and 70 rd. wide. How many bushels of wheat would be taken from it at the average yield of 24 bu. per acre?

21. A rectangular field contains 20 A., and its width is $\frac{1}{2}$ of its length. Find the dimensions in rods.

22. A rectangular field contains 15 A., and its width is $\frac{2}{3}$ of its length. Find the dimensions in rods.

23. A rectangular field contains 90 A., and its length is $\frac{16}{9}$ of its width. Find its dimensions in rods.

24. If 63.39 rd. of fence inclose a circular field containing 2 A., what length will inclose 8 A. in circular form?

25. Which costs the more, to fence in 20 A. in the form of a square, or in the form of a rectangle whose width is $\frac{1}{5}$ of its length?

26. A house is 50 ft. long and 40 ft. wide, with a pyramidal roof 15 ft. high. What is the length of a rafter from the corner of the roof to the vertex of it?

27. The total area of a cube is 294 sq. ft. Find the volume.

28. The total area of a square pyramid is 224 sq. ft., and the base edge is 8 ft. Find the volume.

29. The four faces of a square pyramid are equilateral triangles; the lateral area of the pyramid is $49\sqrt{3}$ sq. in. Find the volume in cubic inches.

30. A dome is in the shape of a hemisphere, which has a radius of 3 ft. 6 in. Find the cost of gilding it at 15 cents per square foot.

31. A kettle is in the shape of a hemisphere; the inner radius is $17\frac{3}{4}$ in., and that of the outer is 18 in. How many gallons will it hold? How many liters? What is the area of the inner surface? Of the outer surface?

32. The radii of three spheres are 3 ft., $2\frac{3}{8}$ ft., and $1\frac{1}{2}$ ft., respectively. If made of lead, and melted into one spherical ball, what would be the radius of the ball in feet?

33. They would make how many spherical bullets, each a quarter of an inch in diameter? (Ex. 32.)

34. How many pounds of iron in the kettle mentioned in Ex. 31? How many Kg. does it weigh?

35. A silver dollar is $1\frac{1}{2}$ in. in diameter (approx.). What is the radius of the circle in which three of these dollars may be placed so as to touch each other, and touch the circumference of the circle?

36. An electric light is 8′ 8″ from the ground. If a man 5′ 4″ in height stands 12′ 6″ from a vertical line through the light, how long is his shadow?

37. The areas of two squares differ by 44 sq. yd., and the lengths of their sides differ by 6 yd. Find their areas.

EXERCISES

I. A boiler house is to be built.

1. How many loads of earth must be removed to make the excavation, which is to be 60 ft. long, 36 ft. wide, and 6 ft. deep? (A load is usually a cubic yard.)

2. The walls of the house are to be 1 ft. thick and 18 ft. high. Allowing for two doors, one 9 ft. by 6 ft., the other 9 ft. by 3 ft., how many bricks are required to build these walls if a brick occupies a space 8 in. by 4 in. by $2\frac{1}{4}$ in.?

3. The comb-shaped roof runs lengthwise, and is $13\frac{1}{2}$ ft. high above the walls. If the rafters project 1 ft. over the walls, how long are they? How many rafters are required if 2 ft. apart? How many board feet in them?

4. How many board feet in the sheathing necessary for this roof?

5. How many thousand shingles are required to cover this house, if they project 3 in. over the ends of the rafters, average 5 in. in width, and are laid so as to allow an exposure of 5 in.? (A double row to start with.)

II. A rectangular lawn 72 ft. long and 42 ft. wide is to be made.

1. The native soil to the depth of 5 in. must be removed. How many wagon loads of this soil are there?

2. How many wagon loads of clay must be obtained to cover the ground to the depth of 7 in.?

3. How many square yards of sodding are required to cover the clay?

4. How many fence posts, 6 ft. apart, are required to fence in this lawn? If 8 ft. long and 4 in. by 4 in., how many board feet in them?

5. If the posts are set flush with the boundary line of the lawn, how many yards of wire are required to make a wire fence 6 wires high?

6. If a cement walk is laid around this lawn just outside of it, how many square yards are there in it, if it is 6 ft. wide?

7. Just outside of this walk there is a narrow strip next to the curbing, 3 ft. wide, to be covered with sod. How many square yards are necessary?

8. What length of curbing (in yards) is required to surround the whole?

III. The Terre Haute Water Works Company has recently built a settling basin, through which the water passes after it leaves the pumps and before it enters the filters.

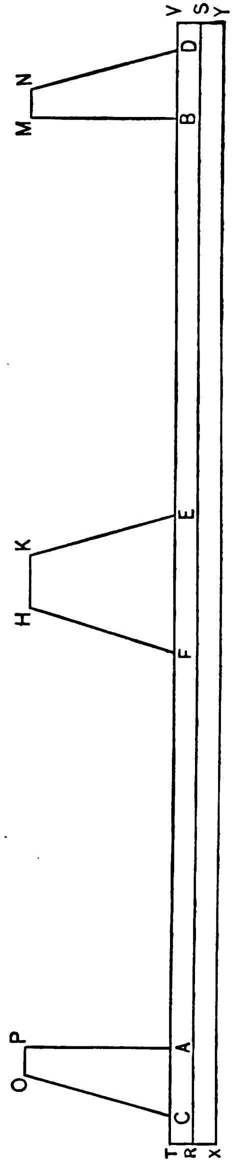
The figure represents a cross-section. The basin is 230 ft. long and 66 ft. wide in the clear. It is 66 ft. from A to B.

$ACOP$ and $BDNM$ represent sections of the walls which surround the entire basin. These cross-sections are 4 ft. at the bottom and 2 ft. at the top, and are 7 ft. high. PA and MB are perpendicular to the floor, and are 7 ft. long. $AC = BD = 4$ ft. $PO = MN = 2$ ft.

$EFHK$ represents a cross-section of a center wall running the entire length, 230 ft., and the base EF is 6 ft., the top HK is 3 ft., and the height is 7 ft.

These surrounding and central walls are made of concrete, and they all stand on a concrete floor 1 ft. thick. This floor, $TRSV$, extends 2 ft. beyond the base of the walls. $TC = DV = 2$ ft.

Hence, $TV = 78$ ft., and the full *length* of the floor is 230 ft. + 4 ft. + 4 ft. + 2 ft. + 2 ft. = 242 ft. This concrete floor rests on a floor of clay, $RSYX$, which is 1 ft. thick.



1. Since the top of the surrounding wall is flush with the surface of the ground, how many cubic yards of earth must be removed to make the excavation for this basin?

2. How many cubic yards of clay are in the clay floor?

3. The clay cost 15¢ per load of $1\frac{1}{4}$ cu. yd. To haul this clay 6 teams and wagons were hired at $3 per day for each team and wagon, each team to haul 16 loads per day. To load and unload these wagons 6 men were employed, 5 of them at $1.50 and 1 man at $1.75 per day. What did the clay cost delivered at the basin?

4. How many cubic yards of concrete were used to make the concrete floor and the surrounding and central walls?

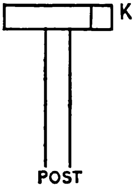
5. The concrete used weighed 140 lb. per cubic foot. What is the total weight of the concrete used?

6. What is the specific gravity of this concrete?

7. How many gallons of water will this basin hold?
(Do not forget the central wall.)

8. What is the weight of the water in the basin when full?

9. For supporting the roof 28 posts, each 11 ft. 8 in. long and 8 in. square, and 14 posts, each 10 ft. long and 8 in. square, were used. How many board feet in these posts?



10. On top of these posts, and just under the horizontal beams, are placed 42 corbels, each 3 ft. long and 8 in. square. How many board feet in them?

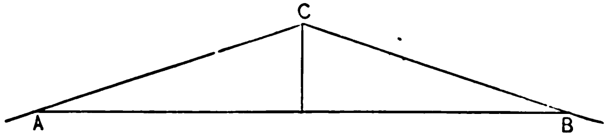
K is a *corbel.*

11. Resting on top of the corbels are 3 beams 10 in. by 8 in., and 2 beams 10 in. by 6 in., all 232.46 ft. long. How many board feet in these beams?

12. The rafters consist of 237 pieces, each 20 ft. long, 10 in. wide, and 2 in. thick; and 237 pieces, each 18 ft. long, 10 in. wide, and 2 in. thick. How many board feet in the rafters?

13. The crest or comb of the roof is 10 ft. higher than the eaves, and the eaves are 70.6 ft. apart. Thus, *A* and *B* are the

N



eaves and C the crest of this cross-sectional view of the roof. Find the length of the slant or slope of the roof.

14. Lumber 1 in. thick was used for sheathing to lie on the rafters and under the metal roof. This sheathing lacked .3 ft. of coming to the eaves. If the roof is 232.46 ft. long, find the number of board feet in the sheathing.

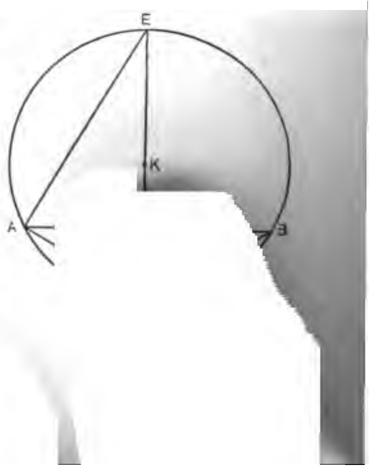
15. How many square yards of metal in the roof?

16. How many square yards of metal were used in making the two gutters under the eaves, being semicircular in form, 8 in. in diameter, and 232.5 ft. long?

17. The gutters under the eaves turn the rainwater which falls on the roof into the basin. How many gallons of water would the basin receive as the result of a rainfall of 1.24 in.?

CHORDS OF CIRCLES

Let AB be any chord of the circle whose center is K. The diameter DE is drawn perpendicular to the chord AB. Draw AD and BD. DC is the height of the arc ADB, and CE is the height of the arc AEB. In geometry, it is shown that DE bisects the chord AB and the two arcs ADB and AEB. Hence, AD or DB is the chord of half the arc ADB, and AE is the chord of half the arc AEB. In geometry, it is further shown that:



$$DC \times DE = AD \times AD, \qquad (1)$$

and
$$DC \times CE = AC \times AC. \qquad (2)$$

From the *former* equation, three very important problems may be stated:

I. Having given the diameter of a circle and the height of an arc, to find the chord of half the arc.

II. Having given the diameter of a circle and the chord of half an arc, to find the height of the arc.

III. Having given the chord of half an arc of a circle and the height of the arc, to find the diameter of the circle.

From the *latter* equation, three more very important problems may be stated:

IV. Having given the diameter of a circle and the height of an arc, to find the chord of the arc.

V. Having given the diameter of a circle and the chord of an arc, to find the height of the arc.

VI. Having given the chord of an arc and the height of an arc, to find the diameter of the circle.

EXERCISES

1. The diameter of a circle is 25 in., and the height of an arc is 9 in. Find the chord of half the arc.

SOLUTION. From the drawing, $DE = 25$ in., and $DC = 9$ in. Hence, $DA \times DA$, or $\overline{DA}^2 = 25 \times 9$, and $DA = 15$ in.

2. The diameter of a circle is 18 in., and the chord of half an arc is 12 in. Find the height of the arc.

SOLUTION. $DE = 18$ in., and $DA = 12$ in. Hence, $DC \times 18 = 12 \times 12$. $\therefore DC = \frac{12 \times 12}{18} = 8$ in.

3. The chord of half an arc is 8 ft., and the height is 5 ft. Find the diameter of the circle.

4. The diameter of a circle is 60 m., and the height of an arc is 40 m. Find the chord of half an arc.

SOLUTION. From the drawing, $ED = 60$ m., and $EC = 40$ m. But $ED \times EC = \overline{AE}^2$. Hence, $\overline{AE}^2 = 60 \times 40$. $\therefore AE = \sqrt{2400} = 20\sqrt{6} = 48.988+$ m.

5. The chord of half an arc is 12 dm., and the diameter of the circle is 20 dm. Find the height of the arc.

6. The chord of half an arc is 16.5 in., and the height of the whole arc is 6 in. Find the diameter of the circle.

7. The diameter of a circle is 30 cm., and the chord of an arc is 18 cm. Find the height of the arc.

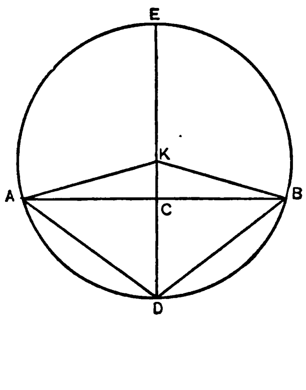
8. The chord of an arc is 10 dm., and the height of the arc is 5 dm. Find the diameter of the circle.

9. The chord of an arc is 28 in., and the height of the arc is 18 in. Find the diameter of the circle.

10. The diameter of a circle is 42 links, and the height of an arc is 10.5 links. Find the chord of the arc.

SEGMENT OF A CIRCLE

In the circle to the left, whose center is K, the chord AB divides the circle into two segments. The chord AB and the arc ADB bound one segment, while chord AB and arc AEB bound the other; they are known as minor and major segments, respectively.



It is difficult to find the area of a segment of a circle without the aid of trigonometry. By the use of trigonometry, the radius of the circle and the angle of

the sector corresponding to the segment may be used to advantage. However, the following rule, taken from Todhunter's "Mensuration for Beginners," can be used when the chord of the arc of the segment and the height of the arc are given.

RULE. *Add together one-fourth of the square of the chord and two-fifths of the square of the height, and multiply the square root of the sum by four-thirds of the height.*

Todhunter says this rule is not exact, but that the error is very small, especially when the angle of the corresponding sector is small.

EXERCISES

1. The chord is 16 in., and the height is 5 in. Find the area of the segment.

SOLUTION. $(16)^2 = 256$. $\frac{1}{4}$ of $256 = 64$. $5^2 = 25$. $\frac{2}{5}$ of $25 = 10$. $64+10=74$. $\sqrt{74} = 8.6+$. $\frac{4}{3}$ of $5 = 6\frac{2}{3}$. $8.6+ \times 6\frac{2}{3} = 57.33+$. $\therefore$ area $= 57.33$ sq. in.

2. The diameter of a circle is 45 in., and the height of an arc is 10 in. Find the area of the segment bounded by the arc and its chord.

NOTE. First find the length of the chord by a previous problem, then find the area.

3. A cylindrical oil tank is 27 ft. long and 77 in. in diameter. It lies in a horizontal position. If the greatest depth of oil in the tank is 15 in., how many gallons of oil are in the tank?

SIMILAR SOLIDS

The base of a rectangular prism is 3 in. long and 2 in. wide, and the altitude is 4 in.

The base of another rectangular prism is 6 in. long and 4 in. wide, and the altitude is 8 in.

Each of the dimensions of the latter is twice the corresponding dimensions of the former. These two prisms are *similar*. The length, width, and height of a rectangular solid are 5 in., 4 in., and 7 in. respectively; those of another are 15 in., 12 in., and 21 in. These two solids are *similar*, because the ratio of any two of the corresponding dimensions is constant = 3.

Any two *rectangular* solids whose corresponding dimensions are proportional are *similar*.

The dimensions of a rectangular solid are 8 in., 6 in., and 10 in. respectively; and those of another are 4 in., 3 in., and 5 in. Are these solids similar? Why? What is the ratio of any two corresponding dimensions? What are their respective volumes? What is the ratio of their volumes? Notice that *the ratio of their volumes is the cube of the ratio of any two corresponding dimensions.*

Hence, if the volume of one of two similar rectangular solids is known, to find the volume of the other, multiply the volume of the known one by the cube of the ratio of any two corresponding dimensions of the unknown and the known one.

Thus, the dimensions of a rectangular solid are 3 ft., 5 ft., and 8 ft., and its volume is 120 cu. ft. If each of the dimensions of another rectangular solid is 4 times the corresponding dimension of the first one, its volume is 64 times the volume of the first one, or 120 cu. ft. $\times 64 = 7680$ cu. ft. Check this result.

A triangular right prism has an altitude of 12 ft., and the sides of the base are 9 ft., 7 ft., and 4 ft. long respectively. Another triangular right prism has an altitude of 36 ft., and the sides of the base are 27 ft., 21 ft., and 12 ft.

These prisms are similar. Why?

Find their volumes and compare them.

The ratio between any two corresponding lines of these solids, as their altitudes, is 3. Is the volume of the second one 3^3 or 27 times that of the first? If so, what law as to the volumes of similar solids is again exemplified?

If two *cylinders of revolution* are such that the altitude of the one is 5 times that of the other, and the radius of the base of the one is 5 times that of the other, the two cylinders are *similar*, because their corresponding dimensions are proportional. The volume of the first one is 125 times that of the second.

Check this statement by using the following cylinders: The altitude and the radius of the base of one are 10 in. and 7 in. respectively, and the altitude and radius of the base of the other are 50 in. and 35 in. respectively.

If the altitude and the radius of the base of one circular cylinder are 5 in. and 2 in. respectively, and the altitude and radius of another cylinder are 15 in. and 6 in. respectively, are these cylinders similar? Why?

Any two *right pyramids* are similar if their bases are similar polygons and the ratio of any two corresponding sides of their bases equals the ratio of their altitudes. Thus, the altitude of a square pyramid is 5 ft. and each side of the base is 3 ft., and the altitude of another square pyramid is 30 ft. and each side of the base is 18 ft.

The volume of the first is 15 cu. ft., and the volume of the second is 216 times 15 cu. ft. Check this last statement. Is the law concerning the volumes of similar solids again sustained?

Any two *cones of revolution* are similar provided their altitudes have the same ratio as the radii of their bases. Thus, the altitude and radius of the base of a cone are $13\frac{1}{3}$ in. and $3\frac{1}{2}$ in. respectively, while the altitude and the

radius of the base of another cone are 40 in. and $10\frac{1}{2}$ in. respectively. These cones are similar. Why? Find out whether the law concerning the volumes of similar solids is here sustained.

All spheres are similar solids. Why?

EXERCISES

1. Given a rectangular prism whose dimensions are 5 in., 6 in., and 7 in. respectively, and another whose dimensions are $12\frac{1}{2}$ in., 15 in., and $17\frac{1}{2}$ in. respectively. Are they similar? Why?

2. What is the ratio of the volumes of the prisms mentioned in Ex. 1? How does this ratio compare with the ratio of the first two corresponding dimensions?

3. The three dimensions of a rectangular solid are 10 m., 12 m., and 15 m. A dimension of this solid 10 m. long corresponds to a dimension 2 m. long of a similar solid. Find the volume of the second solid.

4. The altitudes of two similar triangular right prisms are 3 in. and $13\frac{1}{3}$ in. respectively. Compare their volumes.

5. Two similar hexagonal right prisms have regular hexagons for their bases. If the side of the base of the one is 8.5 dm. and that of the other is 17 cm., compare their volumes.

6. Compare the lateral edges of the prisms mentioned in Ex. 5.

7. Two similar cylinders of revolution have for the radii of their bases lines 6 in. and 7 in. long respectively. Compare their volumes and their altitudes.

8. If two similar right pyramids have square bases, and the side of the base of the one is 21 Dm. and that of the base of the other is 21 dm. long, compare their altitudes.

9. Compare the areas of the bases in Ex. 8.

10. Compare the volumes of the pyramids mentioned in Ex. 8.

EXERCISES

1. The lengths of two similar rectangles are 6 ft. and 9 ft. respectively, and the width of the former is 5 ft. Find the width of the latter.

2. Two triangles are similar; the base and altitude of the one are $17\frac{1}{3}$ dm. and $7\frac{1}{2}$ dm. respectively. If the altitude of the former is 8 Dm., find the altitude of the latter.

3. Two farms are similar in form, and contain 25 A. and 64 A. respectively. One side of the former is 72 rd. Find the corresponding side of the latter in rods.

4. If a pyramid which is 6 ft. high contains 45 cu. ft., find the height of a similar pyramid which contains 120 cu. ft.

5. If a stack of hay which is 6 ft. high weighs 400 lb., find the weight of a similar stack which is 10 ft. high. Also of a similar stack 16 ft. high.

6. How far from the base of a cone whose altitude is 12 ft. must it be cut by a plane parallel to the base that the volume of the frustum will contain $\frac{1}{2}$ as many cubic feet as the cone?

7. If 40,960 A. are represented on a map by an area of 16 sq. in., find the scale of the map in miles to the inch.

8. How many circles, each having a radius of 5 in., are together equal in area to a circle having a radius of 30 in.?

9. How many equilateral triangles, each having a side $7\frac{1}{2}$ in. long, are together equal in area to an equilateral having a side $22\frac{1}{2}$ ft. long?

LATITUDE AND LONGITUDE

The *equator* is a line passing around the earth midway between the north and south poles.

A *meridian* is the boundary line of a circle of the earth which passes through the poles.

The *latitude* of a place is its distance north or south of the equator.

The *longitude* of a place is its distance east or west of a certain meridian.

The *distance* between two places in either case is measured along a curved line, and the units used are degrees, minutes, and seconds.

Since the circumference of the earth contains 360°, the greatest latitude a place can have is 90°, and the greatest longitude is 180°.

Since there is but one equator from which to reckon latitude, the latitude of a place is constant. There are many meridian lines, and since the longitude of any place may be reckoned from any one of them, confusion can be avoided only by selecting one meridian from which the longitude of all places shall be reckoned.

In the world's history most countries have used the meridians through their capitals as their *prime* meridians. France used the one through Paris, Germany the one through Berlin, and the United States the one through Washington, D.C.

In 1844 there was held at Washington, D.C., an International Conference of all the leading countries of the world for the purpose of determining upon a single meridian from which to count or reckon longitude.

A resolution was adopted providing that longitude shall be reckoned from a meridian through Greenwich, England.

The earth makes one complete revolution on its axis every 24 hr., and, since the circumference of the earth contains 360°, this number of degrees passes under the sun every 24 hr. Hence, 15° pass under the sun in an hour. $15° = 60' \times 15 = 900'$. If 900′ pass under the sun's rays in 1 hr. of 60 min., 15′ of longitude will pass in 1 min. $15' = 60'' \times 15 = 900''$. If 900″ pass in 1 min. of 60 sec., 15″ will pass in 1 sec. of time.

∴ 360° of longitude correspond to 24 hr. of time,
15° of longitude correspond to 1 hr. of time,
15′ of longitude correspond to 1 min. of time,
15″ of longitude correspond to 1 sec. of time,
1° of longitude corresponds to 4 min. of time,
1′ of longitude corresponds to 4 sec. of time.

Hence, two places which are 180° of longitude apart have a difference in time of 12 hr.; two places which have a difference in longitude of 120° have a difference in time of 8 hr.; and two places which differ by 18° in longitude differ in time by 1 hr. and 12 min.

Since the earth turns from *west* to *east*, if the sun is over the meridian of a place at any given time it will be over the meridian of a place 60° west of it 4 hr. afterward; if 2° west of it, 8 min. afterward.

If the difference between A and B is 75° 45′ 30″ of longitude, the difference in time is 5 hr. 3 min. 2 sec.,

for 75° corresponds to 5 hr. of time, 45′ corresponds to 3 min. of time, and 30″, to 2 sec. of time.

Hence, a short rule: To find the difference in time between two places when the difference in longitude is given, divide the difference in longitude by 15, and call the result hours, minutes, and seconds.

Since	1 hr.	of time corresponds	to	15°	of longitude,
	2 hr.	of time correspond	to	30°	of longitude,
	7 hr.	of time correspond	to	105°	of longitude,
	20 min.	of time correspond	to	5°	of longitude,
and	40 sec.	of time correspond	to	10′	of longitude,

to find the difference in longitude between two places when the difference in time is given, multiply the difference in time by 15, and call the result degrees (°), minutes (′), and seconds (″) of longitude.

EXERCISES

1. If the difference in longitude between A and B is 67° 43′ 45″, find the difference in time.

2. If the difference in time between A and B is 3 hr. 14 min. 8 sec., find the difference in longitude.

3. When it is 2 P.M. at New York, what time is it at a place 30° 30′ 15″ east of New York?

SOLUTION. 30° 30′ 15″ of longitude corresponds to 2 hr. 2 min. 1 sec. of time. Hence, the time is 2 hr. 2 min. 1 sec. after 2 P.M., or 4 hr. 2 min. 1 sec. P.M., or 2 min. 1 sec. past 4 P.M.

4. When it is 7.45 A.M. at Cincinnati, Ohio, what time is it at K, a place 62° 43′ 20″ west of Cincinnati?

5. When it is 9.23 A.M. at New York (74° 0′ 3″ W.), what is the time at Chicago (87° 37′ 30″ W.)?

6. When it is 2.50 P.M. at Berlin (13° 23′ 53″ E.), what is the time at New York? At Chicago?

7. When it is noon at Washington (77° 2′ 48″ W.), what is the time at Greenwich?

8. What is the longitude of St. Louis, if the difference in time between New York and St. Louis is 1 hr. 5 min. 1 sec.?

9. The difference in time between Philadelphia and Chicago is 49 min. 50 sec. What is the longitude of Philadelphia?

10. Cleveland is 81° 40′ 30″ W., and St. Paul is 93° 4′ 55″ W. When it is 8.10 P.M. at Cleveland, what is the time at St. Paul?

11. Constantinople is 28° 59′ 14″ E., and Quebec 71° 13′ 45″ W. What is the difference in longitude? When it is 4 P.M. at Quebec, what is the time at Constantinople?

12. Find the longitude of a place whose time is 6 o'clock A.M., when it is 4.15 A.M. at Washington, D.C.

13. Boston is 71° 3′ 30″ W., and London is 0° 5′ 48″ W. When it is 8 P.M. at London, what is the time at Boston?

14. Explain the statement, "Mr. Cook delivered a lecture at London, and the people of Boston, his home, read it 3 hr. before he delivered it."

15. A man who had correct time at Chicago traveled along a parallel until his chronometer indicated that it was 1 hr. 4 min. 12 sec. slow. In what direction had he traveled and how far?

16. A man who carried correct time at St. Louis traveled north 5° 21′ 18″. By how much should he change his timepiece, and which way?

17. When it is noon at St. Paul it is 37 min. 12 sec. past 1 P.M. at Bangor. Find the longitude of Bangor.

18. When it is 9 P.M. at Calcutta (88° 20′ E.) it is 27 min. $19\frac{2}{3}$ sec. past 5 P.M. at Jerusalem. What is the longitude of Jerusalem?

19. When it is 4.25 P.M. at Rome (12° 27′ E.) it is 9 hr. 4 min. 24 sec. A.M. at Mobile, Alabama. Find the longitude of Mobile.

20. The local time at Sandy Hook is 4 hr. 56 min. 4 sec. slower than at Greenwich. Find the longitude of Sandy Hook.

21. Chicago, Ill., is 41° 53′ 48″, and Mobile, Ala., is 30° 41′ 26″ north latitude. How many Km. are they apart?

22. St. Petersburg is in 59° 56′, and Constantinople is in 41° 1′ north latitude. How many Km. is the latter south of the former?

STANDARD TIME

The time considered in the preceding problems is the *local* time of the various places. Nearly all of the railroads, cities, and towns of the United States now use the time of some particular meridian. The meridians used are those which are 75°, 90°, 105°, and 120° west of Greenwich, England. The plan was adopted by the railways of the United States in 1883 primarily for their own convenience.

Those places which lie within about $7\frac{1}{2}°$ east or west of the 75th meridian west use the local time of the 75th meridian. Those within about $7\frac{1}{2}°$ east or west of the 90th west use the time of the 90th, and so on for the 105th and 120th meridians.

The local time of the 75th meridian west is Eastern Time.
The local time of the 90th meridian west is Central Time.
The local time of the 105th meridian west is Mountain Time.
The local time of the 120th meridian west is Pacific Time.

Hence, there is just 1 hr. between Eastern and Central Time, 1 hr. between Central and Mountain Time, and 1 hr. between Mountain and Pacific Time.

The lines of division between any two standards is not *exactly* $7\frac{1}{2}°$ on either side of a given meridian, since the scheme was fixed primarily for the railways. The cities at which the railways change from Eastern to Central Time are Buffalo, N.Y.; Pittsburg, Pa.; Wheeling and Huntington, W. Va.; Atlanta and Charleston, S.C. Between Central and Mountain Time, Mandan, N. Dak.; North Platte, Neb.; Dodge City, Kan.; and El Paso, Tex. El Paso has the same time as Charleston.

INTERNATIONAL DATE LINE

If one should start from Indianapolis at noon on Monday, and travel westward just as fast as the sun appears to move, this traveler would arrive at Indianapolis at noon on Tuesday, and would not have traveled during any night. He would be one day behind in his time reckoning. When and where did the change take place whereby he lost a day?

In traveling eastward from Indianapolis at the same rate as the sun appears to travel, the traveler, on arriving at Indianapolis, would have gained a day, apparently.

To save confusion of dates to travelers, there should be some line determined upon at which those who travel eastward should add a day and those who travel westward should subtract a day, in reckoning their time. Such a line is called a *date line*, and if it is agreed upon by the leading nations of the world this line becomes an *international date line*.

Theoretically the 180th meridian is the international date line, and "when regular steamship lines were established on the Pacific Ocean" the change of date was made on crossing the 180th meridian.

However, the 180th meridian has not been agreed upon as the Date Line. In fact, it is an irregular line in the Pacific Ocean extending from north to south. The irregularity in this line is caused by the currents of civilization in making discoveries and settlements.

Magellan discovered the Philippines in 1521, and came from Spanish America in a westward direction to these islands. Hence, the dates established on these islands were one day behind what they should have been.

Other islands in the Pacific were discovered by travelers who came from Europe by the way of the Cape of Good Hope, and hence they established dates which were different from Magellan's by one day. This state of affairs existed among the islands of the Pacific for some centuries. Finally the discrepancy became so inconvenient to the welfare of the Philippines that, as late as 1844, the Governor-General of the Islands decreed "that, for this year only, Tuesday, Dec. 31, be suppressed, and that the day following Monday, the 30th of the same month, be styled Wednesday, Jan. 1, 1845." *Scottish Geo. Mag.*

Beginning at the North Pole the Date Line passes through the Bering Strait, and by most map-makers is shown to pass in close proximity to the coasts of Asia, China, and Japan, curving to the west of the Philippines so as to place them under the same time as the United States; thence to the east of New Guinea and New Zealand, crossing the 180th meridian, passing Chatham Island, and toward the South Pole.

The most western point reached by this irregular line is in its curve to the west of the Philippines, where it touches the meridian of 117° E. The most eastern point is said to be 168° W. Hence, the deviation from a meridian line is about 75°.

PERCENTAGE

$\frac{1}{20} = \frac{5}{100} = .05.$ $\frac{1}{50} = \frac{2}{100} = .02.$

$\frac{3}{25} = \frac{12}{100} = .12.$ $\frac{1}{30} = \frac{3\frac{1}{3}}{100} = .03\frac{1}{3}.$

$\frac{1}{16} = \frac{6\frac{1}{4}}{100} = .06\frac{1}{4}.$ $\frac{3}{14} = \frac{21\frac{3}{7}}{100} = .21\frac{3}{7}.$

The student is familiar with such reductions as those indicated.

The term *per cent* means hundredths; hence, $.05 = 5$ per cent; $.02 = 2$ per cent; $.06\frac{1}{4} = 6\frac{1}{4}$ per cent. The symbol % stands for per cent. Therefore 2 per cent $= 2\%$, and $6\frac{1}{4}$ per cent $= 6\frac{1}{4}\%$. From this, $\frac{4}{25} = \frac{16}{100} = .16 = 16$ per cent $= 16\%$, and $\frac{5}{14} = .35\frac{5}{7} = 35\frac{5}{7}$ per cent $= 35\frac{5}{7}\%$. In short, $\frac{3}{4} = 75\%$; $\frac{4}{5} = 80\%$; $\frac{5}{6} = 83\frac{1}{3}\%$; $3\frac{1}{2} = 3.50 = 350\%$; and $2\frac{2}{3} = 2.66\frac{2}{3} = 266\frac{2}{3}\%$.

The student who can perform the four fundamental operations upon simple numbers, upon common and decimal fractions, and upon compound numbers; who can reduce a fraction of one denomination to that of another, especially common to decimal fractions and *vice versa*, and who has *fair judgment*, can pursue profitably the study of percentage, for there are no new operations to be performed. *Per cent* and its symbol, %, are new.

EXERCISES

1. Express as hundredths and then as per cent: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{2}{7}$, $\frac{3}{8}$, $\frac{5}{7}$, $\frac{7}{8}$, $\frac{4}{4}$, $\frac{6}{11}$, $\frac{5}{12}$, $\frac{7}{5}$, $\frac{16}{9}$, and $\frac{100}{3}$.

2. 5 bu. is what per cent of 8 bu.?

SOLUTION. 5 bu. = $\frac{5}{8}$ of 8 bu. = .62$\frac{1}{2}$ of 8 bu. = 62$\frac{1}{2}$% of 8 bu.

3. A's age is 40 years and B's age is 50 years. The difference of their ages is what *part* of A's age? How many *hundredths?* How many per cent?

4. In Ex. 3, the difference of their ages is what *part* of B's age? What per cent?

5. What is the essential difference between Exs. 1 and 3?

6. On removing from one city to another it was observed that beefsteak was 10¢ per pound in the latter as against 12$\frac{1}{2}$¢ per pound in the former. The difference in price was what per cent of the former?

SOLUTION. The difference $= 2\frac{1}{2}$¢ $= \dfrac{2\frac{1}{2}}{10}$ of 10¢ $= \frac{1}{4}$ of 10¢ $= 25\%$ of 10¢.

7. In Ex. 6, the difference in price was what per cent of the price of the latter?

8. A city lot was bought for \$2000 and sold for \$2250. The profit was what per cent of the cost?

9. If the lot mentioned in Ex. 8 had been bought for \$2250 and sold for \$2000, the loss would have been what per cent of the cost?

10. In a town whose population was 121,600 there were born in a year 7752 children. The number of births was what per cent of the population?

EXERCISES

1. Reduce to hundredths and then to lowest terms: 2%, 4%, 7%, 12%, 29%, $\frac{1}{2}$%, $\frac{3}{4}$%, 1$\frac{7}{8}$%, and 138%.

2. What is 20% of 750 bu.? Of \$225? Of 143 cents?

3. What is $12\frac{1}{2}\%$ of \$840? Of 216 dm.? Of 328 Kg.?

4. What is $\frac{1}{2}\%$ of 500? Of 19? Of 243?

5. What is 25% of $\frac{8}{9}$? Of $\frac{32}{48}$? Of .004?

6. What is $\frac{1}{8}\%$ of 16? Of 2.35 oz.? Of .448 cm.?

NOTE. $\frac{1}{2}\% = \frac{1}{200}$; $\frac{1}{8}\% = \frac{1}{800}$; $\frac{5}{8}\% = \frac{5}{800} = \frac{1}{160}$.

7. A farmer lost 5% of his 80 sheep. How many did he lose?

8. Is it ever advisable to change the number of per cent to a common fraction in lowest terms before multiplying or dividing? When? Is it advisable to change 14% to $\frac{7}{50}$? Why?

9. If a man's salary is \$180 per month, and his expenses are 60% of his salary, how much will be his expenses in 5 mo.?

10. A firm failed for \$14,000. At one time the creditors received a dividend of 40% of the debt, at another time a dividend of 14% of it. How much did the creditors receive?

11. If the population of a certain city in 1890 was 32,500 inhabitants, and the increase in the next decade was 40% of that number, what was the population in 1900?

EXERCISES

1. What is 4% of \$248? 17% of 910 lb.? $\frac{1}{8}\%$ of 300 Kilos?

2. 12 is what per cent of 48? $7\frac{1}{2}$ is what per cent of 75?

3. $16\frac{2}{3}$ is what per cent of $66\frac{2}{3}$? 16 is what per cent of 4?

4. $7\frac{1}{3}$ is what per cent of 2.2? 525 is what per cent of 75?

5. 13 is 5% of what number? 6 is $7\frac{1}{2}\%$ of what number?

6. 14 is $2\frac{1}{3}\%$ of what number? $14\frac{2}{7}$ is $14\frac{2}{7}\%$ of what number?

7. 15 is 250% of what number? 50 is $333\frac{1}{3}\%$ of what number?

SOLUTION. $250\% = 2\frac{1}{2}$. $15 \div 2\frac{1}{2} = 6$. $\therefore$ 15 is 250% of 6.

8. If a horse cost $564 and was sold at a profit of $16\frac{2}{3}\%$ of the cost, find the profit.

9. If the profit in the sale of a horse was $94, and the rate of profit was $16\frac{2}{3}\%$, find the cost.

SOLUTION. $16\frac{2}{3}\%$ of the cost = $94. $\therefore$ the cost $= \frac{\$94}{.16\frac{2}{3}} = \frac{\$94}{\frac{1}{6}}$ = $564.

10. If a horse cost $564 and was sold at a profit of $94, find the rate per cent of profit.

SOLUTION. $94 is $\frac{1}{6}$ of $564. $\frac{1}{6} = .16\frac{2}{3} = 16\frac{2}{3}\%$.

11. A man invests $4320 in real estate, which sum was 60% of his capital. What was his capital?

SOLUTION. 60% of his capital = $4320. $\therefore$ his capital $= \frac{\$4320}{.60}$ = $7200.

12. A carriage was sold for $658, which was $16\frac{2}{3}\%$ more than the cost of the carriage. Find the cost of the carriage.

SOLUTION. $116\frac{2}{3}\%$ of the cost = $658. $\therefore$ the cost $= \frac{\$658}{1.16\frac{2}{3}} = \$564$.

13. A bankrupt's assets are $23,625, which are 40% of his liabilities. What are his liabilities?

Avoid the pernicious 100% method.

14. The English knot is 6080 ft. It is what per cent of the common mile? The common mile is what per cent of the English knot?

15. $640 increased by a certain per cent of itself is $720. Find the rate per cent.

There are two methods of obtaining the solution. What are they?

16. General Grant's monument and the grounds have cost $600,000. Of this sum, New York has subscribed $560,000, and $40,000 was obtained from other sources. What per cent of the cost did New York subscribe? What per cent did others subscribe?

17. Central Park, New York, contains 879 acres, and the new reservoir in the park contains 107 acres. What per cent of the park does the reservoir cover?

18. The old reservoir in the same park contains 35 acres. The area of the old reservoir is what per cent of that of the new?

19. A fertilizer, called Land Plaster, made at Alabaster, Mich., contains 46 parts of sulphuric acid, 33 parts of carbonate of lime, and 21 parts of water of crystallization. What per cent of the fertilizer is sulphuric acid? Carbonate of lime? Water of crystallization?

20. Washington's monument at Washington, D.C., cost $1,200,000. Grant's monument cost what per cent of the cost of Washington's?

21. The height of Mount Everest is 29,005 ft. That of Pike's Peak is 14,147 ft. The height of the latter is what per cent of the former?

22. The height of Mount Shasta is 14,400 ft. Its height is what per cent of the height of Pike's Peak?

23. The height of Mount Etna is 10,874 ft. Its height is what per cent of the height of Mount Shasta? Of Pike's Peak?

24. The height of Mount Sinai is 6541 ft. Its height is what per cent of the height of Mount Shasta? Of Mount Everest?

25. In the Civil War New York furnished 409,561 white troops, and Indiana 193,748. The number furnished by Indiana is what per cent of those furnished by New York?

26. Maine furnished 64,973 troops. This number is what per cent of the number furnished by Indiana? By New York?

27. Illinois furnished 255,057 troops. Her number is what per cent of the number furnished by Maine? By Indiana?

28. The wealth of the United States in 1902 is estimated at $81,750,000,000. That of the United Kingdom is $59,030,000,000. The wealth of the latter is what per cent of that of the former?

29. The wealth of France in the same year is estimated at $48,450,000,000. The wealth of France is what per cent of that of the United States?

30. The wealth of Germany is estimated at $40,260,000,000, and that of Russia at $32,125,000,000. The wealth of Germany is what per cent of that of Russia?

31. The wealth of the United States is what per cent of that of the United Kingdom? Of that of France? Of Germany? Of Russia?

32. The national debt of the United States in 1902 was $1,105,000,000. That of Germany was $3,255,000,000. The national debt of the United States is what per cent of that of Germany?

33. The national debt of the United Kingdom for the same year was $3,530,000,000, and that of Russia $3,555,000,000. What per cent of Russia's debt is that of the United Kingdom?

34. The debt of France is $6,195,000,000. The debt of France is what per cent of that of the United States? Of that of Germany? Of that of the United Kingdom? Of that of Russia?

35. In the year 1901 the United States produced 720,000,000 bu. of wheat, and Russia produced 336,000,000 bu. The production of Russia was what per cent of that of the United States?

36. In the same year France produced 304,000,000 bu.; India, 240,000,000 bu.; Italy, 136,000,000 bu.; and Hungary, 128,000,000 bu. The production of the United States was what per cent of that of France? India? Italy? Hungary?

37. In the year 1900 the pig iron product amounted to 13,789,242 tons in the United States; to 8,908,570 tons in the United Kingdom; to 8,494,852 tons in Germany; to 2,699,494 tons in France; and to 2,321,000 in Russia. The pig iron produced in the United States was what per cent of that in each of the other countries mentioned?

38. The steel product for the year 1900 was: in the United States, 10,087,322 tons; Great Britain, 4,901,054 tons; Germany, 4,799,000 tons; France, 1,604,046 tons; Russia, 1,194,000 tons. The steel product of the United States for 1900 was what per cent of that of each of the other countries mentioned?

39. In 1901 the population of the United States was estimated at 76,000,000 people. That of New York State was 7,150,000. What per cent of the people of the United States lived in New York?

40. If the population of the earth was, in 1901, 1,465,000,000, what per cent of the people of the earth lived in the United States?

41. In 1898 the amount paid by the United States government for pensions was $ 143,982,017.24. For that year Indiana received $ 10,902,433.06 for her pensioners. Indiana received what per cent of all of the pension money for that year?

42. In the year 1898 the United States produced gold to the amount of $ 64,463,000. The whole amount of gold produced by the world was, for the same year, $ 287,428,600. What per cent of the whole world's production of gold was produced in the United States?

PROFIT AND LOSS

There are three general problems in *profit* and three in *loss*. The three in profit are:

(*a*) Given the cost of an article and the rate per cent of profit, to find the profit.

(*b*) Given the cost of an article and the profit, to find the rate per cent of profit.

(*c*) Given the profit and the rate per cent of profit, to find the cost.

There is *one* direct problem and *two* inverse problems. The operation is that of *multiplication* in the direct problem, and *division* in the inverse problems.

Profit is always reckoned on the cost; hence the process is at once suggested by the nature of the problem.

EXERCISES

1. If a lot cost \$540, and was sold at a profit of 20% of the cost, find the profit.

2. If wheat was bought at 55¢ per bushel, and sold at 75¢ per bushel, find the rate of profit.

3. If the profit per ton on coal was 35¢ at a rate of 20% on the cost, find the cost.

SOLUTION. 20% of the cost = 35¢. $\therefore$ the cost $= \frac{35¢}{.20} = \$1.75$.

4. State the three general problems in *loss*. Which one is direct?

5. If coal cost $2.60 per ton and was sold at a loss of 10% on the cost, find the loss. Find the selling price.

6. If a carriage was bought for $120 and sold at a loss of $25, find the rate of loss.

7. If wine was sold at a loss of 75¢ per gallon, which was at the rate of 15%, find the cost per gallon.

8. If pork was purchased at $9 per barrel and sold at 6¢ per pound, find the rate of gain or loss.

9. A merchant marked his goods at 30% above cost, but he allowed a customer 10% off for cash. What was his rate of profit?

10. A merchant buys wine at $4 per gallon; 20% of it was wasted. How much per gallon must he ask for the remainder that he may make 25% profit on his outlay?

11. A man bought a horse which he sold again at a loss of 15% on cost. If he had received $50 more for him, he would have gained 10% on the cost. Find the cost of the horse.

12. A merchant sells to a customer on 30 days' time a bill of goods at a profit of 60% on cost; but the customer failed, and paid but 75¢ on the dollar of his debts. Did the merchant gain or lose, and what rate per cent?

13. A speculator gained $33\frac{1}{3}$% on $\frac{2}{5}$ of his investment, and lost 5% on the remainder, and his net profits were $630. Would he have gained or lost had he gained 30% on $\frac{2}{5}$ of his investment and lost 5% on the remainder? How much?

14. If a merchant buys goods at 20% below cost, and sells them at 16% above cost, what is his rate of gain?

15. A horse-dealer sold 16 horses for $150 each. On one half of them he gained 25%, and on the other he lost 25% on the cost. How much did he lose on the transaction?

16. If a retailer sold $\frac{3}{4}$ of his goods for what they cost, find his rate per cent of profit.

17. If a retail dealer sold $\frac{2}{3}$ of his goods for what $\frac{3}{4}$ of them cost, find the rate per cent of profit.

18. If Mr. K. paid \$12,000 for flour, and sold it at a profit of $33\frac{1}{3}$ on cost to Mr. M., who sold it to Mr. W. at a loss of 25% on cost to him, who paid the more for his flour, Mr. K. or Mr. W.? What has the \$12,000 to do with this problem?

19. If there was a gain of 15% on tea at \$1.08 per pound, what would be the gain per cent if sold at 99¢ per pound?

20. By selling a horse for \$160 a man lost 10% on the cost price. For how much should he have sold him to gain 10% on cost?

21. If a house sold for \$7992 there would have been a gain of 8%. How much per cent is gained or lost by selling it for \$7511?

22. A man sold 2 lots for \$2400 apiece. On one he gained 25% on cost, and on the other he lost 25% on cost. Find the gain or loss in the transaction.

COMMERCIAL OR TRADE DISCOUNT

Most business houses have list prices of their goods printed, but they do not expect to sell at the price listed. It is customary to allow a certain rate per cent off from the list price. This is called a *trade discount.* This list price serves as a basis from which to make various discounts, according to the *amount* of the sale and other causes.

Frequently there are *double* and even *triple* discounts. Thus, 15% and 10% from list may mean 15% off for the wholesale feature and 10% off from the result for cash, which is at once equal to 15% and 10% off. 15%, 10%, and 5% off illustrates a triple discount.

List prices are made considerably above the prices expected for articles of merchandise, so as to avoid changing the price catalogues when the markets change. The discount varies according to the market price.

EXERCISES

1. When the list price of an article is $20, what is the selling price at a discount of 10% from list price? At a discount of 24%? At a discount of 35%?

2. If the list price is $45 and the discount (double) is 10% and 4%, what is the selling price?

SOLUTION. 10% of $45 = $4.50. $45 – $4.50 = $40.50. 4% of $40.50 = $1.62. $40.50 – $1.62 = $38.88. ∴ the selling price is $38.88.

3. If the list price is \$400 and there is a double discount of 15% and 10%, find the selling price.

SOLUTION. 100% − 15% = 85%. 10% of 85% = 8.5%. 85% − 8.5% = 76.5%. 76.5% of \$400 = \$306. ∴ the selling price = \$306.

4. If the list price of an article is \$240 and a trade discount of 20% and 10% is allowed, find the selling price.

Solve by the method in Ex. 2, and then as in Ex. 3.

5. What single discount is equal to the double discount of 15 and 5%? To the double discount of 20 and 5%? To 30 and 15%?

6. What single discount is equal to each of the following double discounts: 40 and 10%? 10 and 40%? $33\frac{1}{3}$ and 25%? 30 and $16\frac{2}{3}$%?

7. What is the net price of a carriage, listed at \$360, at 25 and 20% off?

8. A double discount of 30 and 20% is equal to a single discount of 44%.

SOLUTION. Now 44% = 30% + 20% − 6%. But 6% = 20% of 30%. Hence, to find a single discount equal to a double discount, add the two discounts and from their sum take their product.

GENERALIZED. Let 30% = R and 20% = r. Then 44% = $R + r - Rr$, which is true for this instance.

Test whether the formula $R + r - Rr$ will stand for the single discount equal to a double discount of 40 and 25%. Let 40% = R and 25% = r.

9. Which is the better discount for buyer, 30 and 20%, or 40 and 10%? 50 and 10%, or 40 and 20%? 50 and 10%, or 30 and 30%?

10. Which is the better for the seller, 20 and 5%, or 10 and 15%? $33\frac{1}{3}$ and 25%, or 42 and $16\frac{1}{3}$%? $16\frac{2}{3}$ and 10%, or 20 and $6\frac{2}{3}$%?

	List Price	Discount	Net Price
11.	\$475	40 and 10%	———
12.	\$62.60	10, 2, and 3%	———
13.	\$280	30 and 30%	———
14.	\$400	10 and $33\frac{1}{3}$%	———
15.	\$333.90	$33\frac{1}{3}$ and 10%	———
16.	\$0.99	$11\frac{1}{9}$ and 30%	———

17. If an article is purchased at a double discount of 30 and 10%, and sold at list price, find the rate per cent of profit.

18. The list price of a bill of goods is \$940. If purchased at a discount of 20 and 5%, find the net cost, paying \$17.60 for storage. If sold at list price, find the rate per cent of profit.

19. A merchant purchased 50 bbl. of pork at \$9 per barrel at a discount of 10 and 10%. He pays for freight \$8.40 and labor \$3.25. If he sells at list price, what per cent profit does he make?

20. A man buys goods listed at \$250 at a discount of 10% and a certain per cent off for cash. If the net cost was \$213.75, what was the per cent discount off for cash?

COMMISSION

A very large portion of the business of every community is in the form of *commission business*. Insurance agents, book agents, merchants, gatherers of poultry, buyers of stock from the farms, of wheat and other products, and real estate men get their pay in the form of a *commission* for work done.

A *commission* is a sum paid an individual or firm for buying or selling property for another. It is reckoned as a certain per cent of the buying or selling price in most instances, yet it is not always an application of percentage, for some commissions are specific sums allowed for work done.

If an agent deals in the stocks or bonds of corporations or governments he is called a *broker*, and receives for his work a sum called *brokerage*, which is reckoned on the par value of the stock or bonds handled rather than the *amount* of the purchase or sale price.

An *agent* is a person who transacts business for another, and is sometimes called a *factor* or a *broker*. The person for whom the agent works is called a *principal*.

A *consignor* is a person who sends goods to another. The goods sent are called a *consignment*. The person to whom the goods are sent is called a *consignee*. In many kinds of business an agent is called a *commission merchant*.

EXERCISES

1. Give several illustrations of commission business.

2. If an agent receives $15 per car load for selling mining powder, and sells 5 car loads, what commission does he receive?

3. If an agent charges 50¢ per ton for selling hay, what commission does he receive for selling 30 tons?

Are Exs. 2 and 3 illustrations of percentage? Why?

4. A commission merchant sells 290 bbl. pork at $7 per barrel on a commission of 2% on sales. What is his commission?

5. An agent received $45 for selling meat at the rate of 3% of the selling price. What was the selling price?

6. A *factor* bought wheat for $750, and received a commission of $18.75 for buying. What was the rate per cent of his commission?

7. State the three general problems in *commission.* Also in *brokerage.*

8. An agent sold for his principal 175 mules at an average price of $60 per mule, charging 3% commission. What was his commission? What sum was subject to his principal's order?

If his principal orders him to buy wheat with the proceeds at 64¢ per bushel after deducting a commission of $2\frac{1}{2}$% for buying, how many bushels of wheat did he buy? What was his last commission?

9. A principal sent his agent in Evansville $2080.80 to invest in flour, his commission being 2% on the amount expended in flour. How many barrels of flour could he buy at $4.25 per barrel?

SOLUTION. 102% of the amount expended for flour = $2080.80. The amount expended = $2080.80 ÷ 1.02 = $2040, and the number of barrels = $2040 ÷ $4.25 = 480.

10. An agent sold 6 reapers at \$150 each, and 12 binders at \$180 each. He paid for transportation \$71, and after deducting his commission he remitted \$2575 to his employer. What was the rate per cent of commission?

11. A factor sold merchandise for \$2460 and invested the proceeds, less his commission, in flour. The commission on the merchandise exceeded that on the flour by \$3. What rate did he charge, the rate being the same in both transactions?

12. A manufacturer of maple sirup sent his agent in Chicago 12 tons of sugar. The Chicago agent paid 50¢ per hundredweight to the railroads and \$3 for labor. He charged a commission at the rate of $2\frac{1}{2}\%$. How much would he remit to his employer if he sold the sugar at 25¢ a pound?

INSURANCE

The amount of business done in insurance in the United States is enormous. It is found in every community. There are Life Insurance, Fire Insurance, Accident, Marine, Tornado, Live Stock, Plate Glass, Steam-boiler Insurance, etc. The most important of these are life and property insurance. Among the life insurance companies of the United States there may be found a single company with assets amounting to about \$380,000,000. There are others whose assets amount to about the same. A single company is known to be paying out upwards of \$75,000 per day for each of the six business days of the week, thus amounting to about \$500,000 per week, each week in the year, and this is all done for the benefit of its policy holders. Not many states have good laws for the protection of its policy holders. New York, Pennsylvania, Massachusetts, Indiana, Wisconsin, and Iowa have good laws.

Insurance is an indemnity against loss of life or property. The *policy* is the document issued by the company, which sets forth the kind and amount of property insured, the name of the insured, and the signatures of the officials of the company. The *policy holder* is the person for whose benefit the insurance is taken. The *premium* is the sum paid to the company for the insurance.

The premium is charged at a certain *rate* on the amount of insurance, but it is not always a *rate per cent.*

There are three general problems in insurance:

(*a*) Given the face of the policy and the rate of insurance, to find the premium.

(*b*) Given the face of the policy and the premium, to find the rate of insurance.

(*c*) Given the premium and the rate of insurance, to find the face of the policy.

The *face* of the policy is the sum named in it.

The exercises given will concern property insurance alone.

EXERCISES

1. A factory valued at \$32,000 was insured for 1 year at full value at the rate of $1\frac{1}{2}\%$. Find the premium.

2. A vessel and cargo valued at \$40,000 were insured for $\frac{5}{8}$ of their value at the rate of 2% per year. Find the yearly premium.

3. A merchant's stock was insured for \$36,000; half of the amount at the rate of $\frac{7}{8}\%$, $\frac{2}{3}$ of the remainder at the rate of $\frac{3}{4}\%$, and the remainder at 1%. Find the total premium paid.

4. A cargo of 4000 bu. of wheat worth 80¢ per bu. was insured at $1\frac{1}{2}\%$ for $\frac{4}{5}$ of its value. If the cargo was lost, how much did the owner receive?

5. What will it cost to insure a factory valued at \$25,000 at $\frac{4}{5}\%$, and the machinery valued at \$12,000 at $\frac{7}{8}\%$?

(Sometimes a policy sufficiently large to cover both the property insured and the premium is taken out.)

6. Find the face of the policy which must be taken out to cover the value of lumber worth \$24,822, and the premium at the rate of $1\frac{1}{2}\%$.

SOLUTION. $98\frac{1}{2}\%$ of the face of the policy = \$24822. The face of the policy $= \frac{\$24822}{.985} = \$25200$.

7. Property is insured for \$12,000 in one company, for \$8000 in a second company, and \$5000 in a third. If a loss of \$5000 should occur, what would each company pay the insured?

8. If a premium of \$48 is paid for insuring property valued at \$4000, find the rate.

9. If a barn was insured for $\frac{4}{5}$ of its value at $2\frac{3}{4}\%$, and the premium paid was \$110, what was the value of the barn?

10. A property was insured for \$6400 at $2\frac{1}{4}\%$ per year. During the fourth year the property was destroyed. Find the actual loss to the company.

TAXES

Every form of government requires money with which to enable it to discharge its obligations. This money is collected from the people who are the subjects of the government, and it is called a *tax*. Taxes are collected by the Federal government, the state government, the county, township, and city governments for their support.

The United States government pays out $140,000,000 per year for *pensions*, $175,000,000 per year for the *army* and *navy*, many millions for internal improvements, other millions for the salaries of the President, congressmen, judges of the United States courts, and others connected with the running of the government. The expenses of the United States government are something over $1,000,000 per day for every day in the year, and this great expense is met by taxation.

The *state* must provide for taking care of the insane, the blind, the deaf, the dumb, and other unfortunates, and the criminals; it supports schools to educate the more unfortunate; it must pay the salaries of the various officials of the state, and look after improvements, all of which requires money. This money is obtained by taxation.

The *county* has need of much money with which to build court-houses, build bridges, educate its people, take care of its poor, establish courts of justice, etc. These

expenses are all met by taxation. The *township* has some expenses to meet which are not met by the county nor by the state.

The *city* must provide for costly sewers, for the maintenance of good streets, for fire protection, for police, for education, for various improvements, etc., the cost of all of which is met by taxation.

State and local taxes are usually obtained by two methods: (*a*) Most states levy a specific tax on each male citizen over 21 and under 50 years of age, of from $1 to $4 each. This is called a *Poll Tax*. (*b*) A tax is levied upon the property of citizens. This tax is called a *Property Tax*, and is usually tabulated at so much per $100 valuation of property. Property tax is obtained from *real estate* and *personal* property.

The *legislature* of a state determines the amount of money to be raised for state purposes. The *county commissioners* or some board of authority determine this sum for the county. The *common council* acts for the city.

An officer called the *Assessor* makes a list of the taxable property in a given district. Such a list is called an *Assessment Roll*. In Indiana such a list is based upon the amount of property on hand April 1st of each year, and is made out as soon thereafter as is convenient.

The tax to be paid according to this list is due the next year. Thus, the taxes for the year 1901 are due in the year 1902; the first half on or before the first Monday in May, and the second half on or before the first Monday in November. Each state makes its own laws in this respect.

Whenever the assessor has made known the property available for taxation, and the proper authorities have determined how much money must be obtained by taxation, then the *rate* is determined.

EXERCISES

1. In the year 1900 the state of Indiana had property listed at $1,321,000,000. The state estimated that for the year 1901 $3,900,000 would be needed for state purposes. If there were 493,752 polls, each 50¢, find the property tax on each $100 worth of property.

2. How much state tax would a citizen pay who had property valued at $3600?

3. For the support of the State Normal School Indiana pays at the rate of $\frac{1}{20}$ of a mill on the dollar. Not allowing for delinquencies, what sum did the school receive in the year 1901?

4. The rate for the support of Indiana's three educational institutions is $1\frac{7}{8}$¢ per $100 valuation of property. How much did they receive in the year 1901?

5. Indianapolis has property assessed at $125,000,000. If the city wishes to raise $1,384,000, and has 18,000 polls, each paying 50¢ poll tax, find the tax on each $100 worth of property. What does Indianapolis pay annually to support the State Normal School?

6. Suppose the street-car plant at Indianapolis is assessed at $3,200,000, what is the annual tax of the company for state purposes? For the three educational institutions mentioned in Ex. 4?

7. If a city has property valued at $11,000,000, and wishes to build a high school building valued at $45,100, what will be the tax on each $100, no allowance for delinquencies?

8. Milwaukee, Wis., had in 1899 property valued at $145,245,750. At 95¢ per $100 valuation, what tax could she raise if she has 20,451 persons paying poll tax at 75¢ each?

9. Richmond, Ind., in 1900 levied a tax of $1.04 per $100 on the property of the city, and realized $124,800 from it. How much property had the city that year?

10. State the three general problems concerned with property tax. The three concerned with poll tax.

11. If Evansville should raise a tax of \$291,191.50 from property at the rate of \$1.15 per \$100, find the value of the city property. Suppose the sewer tax is 11¢ and the city hall tax is 9¢ per \$100 valuation, what is the amount of tax received for both?

12. What does Evansville pay annually for the state benevolent institutions, the rate being 5¢ per \$100?

13. Suppose Terre Haute, Ind., for the year 1901 realized \$226,893.85 by a tax on property at \$1.14 per \$100, after paying 1% commission for collecting. Find the property valuation.

14. In the year 1900 Clinton County, Ind., had property valued at \$18,533,000, and 5062 polls assessed at \$1.50 each. The county levied a tax of 43¢. How much was realized after paying 1% for collection?

15. A man has a farm of 200 acres appraised at \$45 per acre. For taxation the farm was assessed at $\frac{5}{8}$ of its appraised value. If this man pays for state and county purposes \$1.22, and for city purposes \$1.08, and a poll tax of \$2.75, find his annual tax.

16. At $5\frac{1}{2}$ mills on a dollar for property tax, and a poll tax of \$1.25, how much is the total tax of a man who owns a farm of 240 acres assessed at \$20 per acre, and who has personal property valued at \$2132?

INTEREST

Interest is an application of percentage, since the rate is usually expressed in hundredths. Every civilized community is concerned with the borrowing and the lending of money; indeed, it is an evidence of a healthy growth.

Interest, or *usury*, as it was called in very early times, has been practiced since the time of the Babylonians. References to borrowing, lending, and interest or usury are found in the Bible: Ex. xxii. 25; Lev. xxv. 36, 37; Deut. xxiii. 19, 20; Neh. v. 7, 10; Ps. xv. 5; Prov. xxviii. 8; Isa. xxiv. 2; Jer. xv. 10; Ezek. xviii. 8, 13, 17; Ezek. xxii. 12; Matt. xxv. 27; Luke vi. 34; Luke xix. 23. From these references it is seen that at first the making of any charge for the use of money was prohibited under penalty. As civilization grew older, a charge was allowed against a stranger, but not against a brother. Finally it was regarded as just to take interest from any one, as is suggested in Matt. xxv. 27: "Thou oughtest therefore to have put my money to the exchangers, that at my coming I should have received mine own with usury [interest]."

The last view is evidently the correct one. Aristotle contended that money as such could not beget money, hence it was wrong to charge for the use of the same. He neglected to see that lenders of money could easily exchange it for lands, houses, and other forms of property, and thus rent these out for money, or that borrowers could

invest the money borrowed in some form of property, and in this form it would bring increase.

The Greeks did not follow the teachings of the Mosaic Law, nor of Aristotle, for they paid as high as 30% for the use of money.

The Romans in their most flourishing period paid very high rates of interest. However, they at times prohibited it.

The Venetians in the fourteenth century looked upon the defense of interest as heresy.

The history of England is replete with references to the prohibiting and allowing of interest. In 1546 a law was passed allowing 10%, and in 1552 a law was passed prohibiting it, and declared the charging of interest as "contrary to the will of God."

In the latter part of the sixteenth century the word "interest" began to be used exclusively for what had been called *usury*, and the meaning of usury was restricted to a charge in excess of the legal rate of interest. In Indiana any rate above 8% is usury.

The United States is unique among the countries of the world in respect to usury laws. It is the tendency, however, in *this* country now to allow the rate of interest to be settled between the parties concerned. It is difficult to legislate a rate of interest for a whole country, for like conditions do not prevail in all portions of it, nor at all times. The rate for a short-time loan ought to be higher than that for a long-time loan.

Interest calculations should be subject to contracts, to special laws enacted, and to laws which have come from custom without legislation. Voluntary contracts in loaning money should be as free as in other kinds of business. The government should regulate the weight and fineness

of coins just as it regulates weights and measures in other contracts, but the rate of interest should be left to the parties concerned just as the prices of the various commodities are. A liveryman rents his horses and carriages at whatever charges he can maintain in competition with others. The owner of a house and lot can contract with a tenant freely without interference from laws.

The bicycle dealer rents some of his machines at such prices as he sees fit.

The owners of money should enjoy the same privileges.

Primarily the laws against usury were intended to protect the poorer classes. But conditions have changed. The middle and poorer classes are becoming the lenders (see page 237 for report as to the individual deposits in the banks of the country), and the richer classes borrow for speculation.

In the loaning of money, there are four things to consider: the principal, the rate, the time, and the interest. These give rise to four general problems. Let the student state them in language. If $p =$ the principal, $r =$ the rate, $t =$ the time, and $i =$ the interest, let the student give the four formulas corresponding to the four general problems. That the product of the principal by the rate should give the interest for one year is partly natural and partly conventional.

The borrowers in very early times, being engaged in agricultural pursuits, depended upon the return of their crops for the means of paying off their debts, and the crops depended upon the seasons. Let the student think of some conventional reasons why the year is the unit of time.

Hence, when using the time as a factor of interest it must be expressed in years.

SIMPLE INTEREST

Simple interest is the interest on the original principal only, and this is generally understood by the term *interest.*

If the time is not given, it is found by compound subtraction, allowing 12 months to the year, and 30 days to the month. Thus, the time from March 21, 1900, to Sept. 1, 1902, is 2 years 5 months and 10 days.

$$\begin{array}{rrr} 1902 & 9 & 1 \\ 1900 & 3 & 21 \\ \hline 2 & 5 & 10 \end{array} \qquad 2 \text{ yr. } 5 \text{ mo. } 10 \text{ da.} = 2\tfrac{4}{9} \text{ yr.}$$

EXERCISES

1. What is the interest of \$600 for 5 yr. at $5\frac{1}{2}\%$?

SOLUTION. $prt = i.$ $p = \$600$, $r = .05\frac{1}{2}$, and $t = 5$ yr.
$\$600 \times .05\frac{1}{2} \times 5 = \$165.$ $\therefore$ interest $= \$165.$

2. What is the interest of \$450 for 2 yr. 5 mo. at 5% ?

SOLUTION. $prt = i.$ $p = \$450$, $r = .05$, $t = 2\frac{5}{12}$ yr.
$\$450 \times .05 \times 2\frac{5}{12} = \$54.375.$ $\therefore$ interest $= \$54.38.$

3. Find the interest of \$400 for 2 yr. 4 mo. 12 da. at 6%.

SOLUTION. 2 yr. 4 mo. 12 da. $= 2\frac{11}{30}$ yr. $= t.$
$\$400 \times .06 \times 2\frac{11}{30} = \$56.80.$ $\therefore$ interest $= \$56.80.$

4. What is the interest of \$640 from July 6, 1900, to Oct. 1, 1902, at 5% ?

SOLUTION. $$\begin{array}{rrr} 1902 & 10 & 1 \\ 1900 & 7 & 6 \\ \hline 2 & 2 & 25 \end{array} \qquad 2 \text{ yr. } 2 \text{ mo. } 25 \text{ da.} = 2\tfrac{17}{72} \text{ yr.}$$

$\$640 \times .05 \times 2\frac{17}{72} = \$71.55\frac{5}{9}.$ Interest $= \$71.56.$

5. What is the interest of \$340 at 5% from Feb. 20, 1899, to Dec. 30, 1901 ?

6. What principal at 7% will draw $45.50 from April 1, 1899, to July 1, 1902?

SOLUTION.

1902	7	1
1899	4	1
3	3	0

3 yr. 3 mo. = $3\frac{1}{4}$ yr.

$$p = \frac{i}{rt} = \frac{\$45.50}{.07 \times 3\frac{1}{4}} = \frac{\$45.50}{.22\frac{3}{4}} = \$200. \quad \therefore \text{ Principal} = \$200.$$

7. What rate per cent on a principal of $320 will draw $32.80 from Sept. 14, 1899, to Dec. 24, 1901?

SOLUTION.

1901	12	24
1899	9	14
2	3	10

2 yr. 3 mo. 10 da. = $2\frac{5}{18}$ yr.

$$r = \frac{i}{pt} = \frac{\$32.80}{\$320 \times 2\frac{5}{18}} = .04\frac{1}{2}. \quad \therefore \text{ The rate} = 4\frac{1}{2}\%.$$

8. In what time will $250 at 6% draw $52.50?

SOLUTION. $t = \frac{i}{pr} = \frac{\$52.50}{\$250 \times .06} = \frac{\$52.50}{\$15} = 3\frac{1}{2}. \quad \therefore \text{ Time} = 3\frac{1}{2}$ yr.

If the date of the note is April 4, 1899, the interest will equal $52.50 on Oct. 4, 1902.

NOTE. If money is loaned at *simple interest*, no interest can be collected till the principal is collected, and no interest is allowed to draw interest.

9. What is the simple interest of $75 at 7% for 1 yr. 2 mo. 15 da.?

10. What is the simple interest of $214 at 5% for 2 yr. 8 mo. 20 da.?

11. What is the simple interest of $540 at 5% for 2 yr. 3 mo. 9 da.?

12. What is the simple interest of $80.40 at 7% for 1 yr. 10 mo. 20 da.?

13. What is the simple interest of $125.45 at 6% for 9 mo. 10 da.?

What amount is due at settlement, no interest having been paid in the meantime, on a note of:

14. \$ 750 dated March 14, 1898, at 5%, settled Sept. 23, 1900?

15. \$ 225 dated June 21, 1897, at 6%, settled Jan. 1, 1899?

16. \$ 217.50 dated Oct. 1, 1899, at 5%, settled May 19, 1901?

17. \$ 453.14 dated Nov. 9, 1900, at 7%, settled Feb. 1, 1902?

18. \$ 973.35 dated Dec. 16, 1900, at $5\frac{1}{2}$%, settled April 15, 1902?

NOTE. Very few borrowers or lenders are concerned about finding the principal, the rate, or the time when the three other elements are given. The *direct* problem is the one met in business. A few *inverse* problems are here given.

19. In what time will \$ 175 draw \$ 25 at 6%?

20. In what time will \$ 219.50 draw \$ 30.50 at 5%?

21. At what rate will \$ 545 draw \$ 50 in 1 yr. 9 mo. 12 da.?

22. At what rate will \$ 375 draw \$ 60 in 2 yr. 7 mo. 15 da.?

23. What principal will draw \$ 45.25 at 5% in 1 yr. 2 mo. 5 da.?

24. What principal will draw \$ 73.14 at 6% in 2 yr. 4 mo. 15 da.?

25. What amount was due at settlement on a note of \$ 544.35, dated Aug. 11, 1899, and settled May 3, 1901, at 5%?

26. What amount was due at settlement on a note of \$ 235.40, dated March 14, 1900, and settled Dec. 24, 1901, at $5\frac{1}{2}$%?

27. What amount will be due on a note for \$ 560.75, dated Sept. 26, 1900, and settled Jan. 1, 1903, at $5\frac{1}{2}$%?

NOTE. The 25th, 26th, and 27th problems imply that no interest was paid till settlement.

ANNUAL INTEREST

It is a principle of law that, in the absence of any agreement, all money bearing interest, or that is due and not paid, shall draw *simple interest.* It often happens that

contracts are made by which the interest is to be paid annually, semi-annually, or quarterly. When the interest is not paid at the time specified in the contract, such interest draws interest to the time of settlement.

Some courts allow unpaid interest to draw interest until settlement, but many states have laws against it. It is a good custom to draw up a note for the principal without interest for the time, and a note for the unpaid interest of each year. These latter notes are called *interest notes*. By this plan all unpaid simple interest will draw interest.

Interest which is contracted to be paid at the end of each interest year is called *annual interest*.

EXERCISES

1. $800. RILEY, IND., Aug. 14, 1900.

Four years after date, I promise to pay to the order of Higsbee & Co., Eight Hundred Dollars, value received, with interest at 5%, *payable annually*. SILAS JONES.

If no interest is paid till settlement date, how much is then due?

SOLUTION. $800 at 5% in 4 yr. amounts to $960 at simple interest. The first year's interest of $40 will draw interest at 5% for 3 yr., the second year's interest will draw interest for 2 yr., and the third for 1 yr., which is the same as $40 drawing interest for 3 yr. + 2 yr. + 1 yr., or 6 yr., at 5%. The interest on $40 at 5% for 6 yr. is $12.

The amount due 4 yr. hence = $960 + $12 = $972.

NOTE. The result contains three elements: the *principal*, $800; the *simple interest*, $160; and the *annual interest*, $12; or, $800 + $160 + $12 = $972.

2. A note for $1200 was given Sept. 15, 1900, with interest from date at 5%, payable annually. If no interest was paid until settlement, what amount was due then, Nov. 30, 1902?

SOLUTION.	1902	11	30	
	1900	9	15	
	2	2	15	2 yr. 2 mo. 15 da. = $2\frac{5}{24}$ yr.

The simple interest of $1200 at 5% for $2\frac{5}{24}$ yr. = $132.50.

The first year's interest, $60, is on interest	1 yr.	2 mo.	15 da.
The second year's interest, $60, is on interest		2 mo.	15 da.
This is the same as $60 on interest for	1 yr.	5 mo.	

The interest on $60 at 5% for 1 yr. 5 mo. = $4.25.

The amount due Nov. 30, 1902, = $1200 + $132.50 + $4.25 = $1336.75.

3. What is due on the note in Prob. 2 if the interest is to be paid semi-annually?

SOLUTION. The interest of $1200 at 5% for 2 yr. 2 mo. 15 da. = $132.50.

The first 6 mo. interest is on interest for 1 yr. 8 mo. 15 da.

The second 6 mo. interest is on interest for 1 yr. 2 mo. 15 da.

The third 6 mo. interest is on interest for 8 mo. 15 da.

The fourth 6 mo. interest is on interest for 2 mo. 15 da.

This is the same as $30 on interest for 3 yr. 10 mo.

The interest of $30 at 5% for 3 yr. 10 mo. = $5.75.

∴ the amount due Nov. 30, 1902, = $1200 + $132.50 + $5.75 = $1338.25.

4. A note of $450, dated March, 1, 1899, at 6% interest, payable annually, is due March 1, 1903. What amount is due at settlement, nothing having been paid?

5. A note for $360 is dated Sept. 20, 1898, with interest at 6%, payable annually. What will be due Jan. 20, 1904, nothing having been paid?

6. A note for $475 is dated April 6, 1898, with interest at $5\frac{1}{2}$%, payable annually. What was due Nov. 27, 1901?

7. A note of $600 was dated May 14, 1900, with interest at 5%, payable quarterly. What was due July 14, 1902, nothing having been paid?

8. Three years after date I promise to pay Caleb Hughes, or order, $380, with interest at 5%, payable semi-annually, value received. OTIS PETERS.

COLD WATER, MICH., Oct. 4, 1898.

Michigan allows interest on deferred interest payments. If nothing was paid till settlement, what was then due?

COMPOUND INTEREST

Loan and building associations, savings banks, and other financial institutions regard the interest due at the end of each interest period as new principal, and apply it as such. Thus, interest draws interest, and soon, interest of interest draws interest. The interest thus accumulated on a given sum is called *compound interest.*

In *annual interest*, interest which comes from the principal, if not paid when due, draws interest, but interest of interest cannot do so.

EXERCISES

1. What is the compound interest of $600 for 4 yr. at 4%?

Solution. The amount of $600 for 1 yr. at 4% = $624.
The amount of $624 for 1 yr. at 4% = $648.96.
The amount of $648.96 for 1 yr. at 4% = $674.92.
The amount of $674.92 for 1 yr. at 4% = $701.92.

∴ At the end of the fourth year the amount due = $701.92.
$701.92 − $600 = $101.92, which equals the compound interest.

2. What is the interest on $600 for 4 yr. at 4% payable *annually*, if nothing is paid till settlement? By how much does the annual interest differ from the compound interest? Account for this exactly.

3. What is the *simple* interest of $600 for 4 yr. at 4%? By what sum does this differ from the *annual* interest?

4. How long after the date of a note does *annual* interest begin to differ from *simple* interest? How long after the date does *compound* interest begin to differ from *annual* interest?

ACCURATE INTEREST

Simple, annual, and compound interests are based upon 30 days to the month, and 12 months, or 360 days, to the year. For short periods this plan is not very accurate.

It is more accurate to account for the exact number of days and allow 365 days to the year.

EXERCISES

1. What is the accurate interest of \$300 from April 4 to Sept. 9 at 5%?

Solution. The exact number of days from April 4 to Sept. 9 $= (26 + 31 + 30 + 31 + 31 + 9)$ da. $= 158$ da. $= \frac{158}{365}$ yr. $\$300 \times .05 \times \frac{158}{365} = \$6.49$. $\therefore$ The accurate interest $= \$6.49$.

2. Find the accurate interest of \$375 at 6% from Aug. 19 to Dec. 24.

3. Find the accurate interest of \$672 at $5\frac{1}{2}\%$ from Oct. 12 to Feb. 23.

4. Find the accurate interest of \$476 at $5\frac{1}{2}\%$ from Jan. 28 to Apr. 5, 1904.

PARTIAL PAYMENTS

If payments are made on notes between the date and the time of settlement, such payments are called *partial payments.* It is customary for the creditor to specify these on the back of the note and sign his name and date to each.

There are several rules of action when partial payments have been made. Among the prominent ones of this country are: the United States Rule, the Merchant's Rule, the Connecticut Rule, the New Hampshire Rule, and the Vermont Rule. Since the Connecticut, New Hampshire, and Vermont Rules concern the respective states after which they are named, attention here will be confined to the United States and Merchant's Rules.

The rule, in most of the states of the Union, for computation of interest on notes where partial payments have been made, is based upon the decision of Chancellor Kent, 2 Johnson's Chancery Reports, p. 209. The language of Chancellor Kent is as follows:

"The rule for casting interest where partial payments have been made is to apply the payment in the first place to the discharge of the interest then due. If the payment exceeds the interest, the surplus goes towards discharging the principal, and the subsequent interest is to be computed on the balance of principal remaining due. If the payment be less than the interest, the surplus of the

interest must not be taken to augment the principal; but interest continues on the former principal until the period when the payments taken together exceed the interest then due, and then the surplus is to be applied towards discharging the principal; and the interest is to be computed on the balance of the principal as aforesaid."

The decision of Chancellor Kent has been adopted by the Supreme Court of the United States, and is formulated as the United States Rule for Partial Payments as follows: Find the simple interest on the face of the note from the date of the note to the date of the first payment. If the payment made is sufficient to discharge the interest thus found, or more, add the interest to the principal and from this sum subtract the payment; the remainder is the second principal. Do in like manner with the second payment, and those following, to the time of settlement. If, however, at any time a payment is *not* sufficient to pay off the interest *then* due, the interest is then calculated to a time when the payment or sum of payments *is* large enough to pay all unpaid interest which may have accrued. From the amount then due subtract the sum of the payments to date, and proceed as before.

The Supreme Court of Indiana, in the case of Wasson *vs.* Gould, 3 Blackford, p. 18, adopted the language of Chancellor Kent as a rule for the computation of interest, and this rule "has never been departed from in Indiana so far as I am advised, and I suppose in most of the other states in the Union it is the same." — U. S. District Judge, John H. Baker.

The practical effect of this rule is to discourage the maker of a note in making any partial payments till time

of settlement. If a payment is made which is less than the interest up to the time it was made, the debtor loses the interest on the payment for a period of time. If a payment is made within a year from date which is large enough to pay off the interest or more, the debtor pays compound interest.

An extreme example will illustrate one objectionable feature of the United States Rule:

If a *debtor* owes $5000 due in 1 yr. at 6%, and pays $20 per month during the year, he will at no time pay enough to discharge the interest for the month. Hence, at the end of the year he will have paid in $240 and still owe $5300 less this $240, or $5060. The $20 per month would, if loaned out, have amounted to $246.60. The *creditor* would thus be profited by $6.60 interest on these payments, for which the *debtor* received no credit.

EXERCISES

1. $350. FT. WAYNE, IND., NOV. 20, 1899.

One year after date, I promise to pay to Henry Ives, or order, Three Hundred Fifty Dollars, at the First National Bank, with interest at 7%, value received. JOHN NIMROD.

Indorsed: March 20, 1900, $75; June 1, 1900, $50; Nov. 20, 1900, $6; May 20, 1901, $80. What was due Oct. 1, 1901?

SOLUTION BY THE UNITED STATES RULE

yr.	mo.	da.
1901	10	1
1901	5	20
1900	11	20
1900	6	1
1900	3	20
1899	11	20

Notice that the periods of time are found by subtracting each from the one above it, and the results are set down as in the second group.

0	4	0	. . .	$75
0	2	11	. . .	$50
0	5	19	. . .	$6
0	6	0	. . .	$80
0	4	11	. . .	Settlement
1	10	11		

Principal	=	$350.000
Interest to March 20, 1900	=	8.166
The amount	=	$358.166
Payment	=	75.000
New principal	=	$283.166
Interest to June 1, 1900	=	3.909
		$287.075
Payment to June 1	=	50.000
		$237.075
Interest to Nov. 20, 1900	=	7.790
The $6 payment does not discharge the interest due at time of payment.		
Interest to May 20, 1901	=	8.298
		$253.163
Payments $80 + $6	=	86.000
		$167.163
Interest to date of settlement	=	4.257
Amount due at settlement	=	$171.420

2. $273.40. GREENCASTLE, IND., July 6, 1900.

Four months after date, I promise to pay to the order of Lemuel Alsap, Two Hundred Seventy-three and $\frac{40}{100}$ Dollars, with interest at 6%, value received. JAMES HEBER.

Indorsed: Oct. 4, 1900, $50; Dec. 31, 1900, $43; March 21, 1901, $64. What was due Sept. 30, 1901?

3. $743.15. PEORIA, ILL., Oct. 6, 1900.

On demand, we promise to pay Gotlittle & Wantmuch, or order, Seven Hundred Forty-three and $\frac{15}{100}$ Dollars, value received, interest at 5%. GOTMUCH & WANTLITTLE.

Indorsed as follows: Dec. 24, 1900, $100; April 1, 1901, $75; Sept. 30, 1901, $150. What was due Dec. 24, 1901?

4. $475. DECATUR, Jan. 1, 1900.

On or before one year after date, I promise to pay John James, or order, Four Hundred and Seventy-five Dollars, value received, with interest at 6%. JAMES JOHN.

Payments as follows: July 1, 1900, $10; Sept. 15, 1900, $8; Nov. 10, 1900, $200. What was due Jan. 1, 1901?

THE MERCHANT'S RULE

A rule for computing interest where partial payments have been made, which is in use among many business men, is called the Merchant's or Mercantile Rule. It has been sanctioned in several legal decisions.

The primary idea of this rule was that it concerned problems in which the date of settlement was not more than a year from date, and all partial payments were made within that time.

Where partial payments are made on a note, it is more equitable than the United States Rule.

RULE. Find the amount of the principal and the interest from date of the note to the date of settlement. Find the amount of each payment and its interest from the date of the payment to the date of settlement. Deduct the sum of the payments and their interests from the amount of the principal. The result is the sum due.

EXERCISES

1. Consider the first exercise of the United States Rule. Find the periods of time by compound subtraction.

Solution

1901	10	1
1899	11	20
1	10	11
1901	10	1
1900	3	20
1	6	11
1901	10	1
1900	6	1
1	4	0
1901	10	1
1900	11	20
0	10	11
1901	10	1
1901	5	20
0	4	11

	yr.	mo.	da.		
The amount of $350 for	1	10	11		= $395.664
The amount of $75 for	1	6	11	= $83.035	
The amount of 50 for	1	4	0	= 54.666	
The amount of 6 for		10	11	= 6.352	
The amount of 80 for		4	11	= 82.037	
The total amount of credits					= 226.090
The amount due Oct. 20, 1901,					= $169.574

The amount due by the U. S. Rule	= $171.420
The amount due by the Merchant's Rule	= 169.574
The difference to the nearest cent	= $ 1.85

It is easy to account for this difference. The interest on the principal to date of first payment is $8.166, which, by the United States Rule, is turned into principal, and draws interest to Oct. 20, 1901.

The interest to the date of the second payment is $3.91, and is turned into principal to draw interest to Oct. 20, 1901. Since the $6 payment was too small to discharge the interest, the debtor lost the use of the $6 from the date it was made to the date of the next payment. The interest which accrued between date of second payment and the date of fourth payment is $16.09, and is turned into principal and draws interest to Oct. 20, 1901. Now, by the United States Rule, the debtor loses the interest on these sums.

			yr.	mo.	da.	
The interest on	$8.166	at 7 % for	1	6	11	= $0.87
The interest on	3.91	at 7 % for	1	4	0	= 0.36
The interest on	6.00	at 7 % for	0	6	0	= 0.21
The interest on	16.09	at 7 % for	0	4	11	= 0.41
The total loss to the debtor,						$1.85

This is the difference between the two results by the United States Rule and the Merchant's Rule.

If a note on which partial payments have been made goes to a bank for collection, unless the contract forbids it, the amount due is found by the United States Rule. However, many bank officials ask the parties to a note to agree to abide by the Merchant's Rule as the more equitable, where payments have been made at short intervals,

but in the absence of agreement the United States Rule is used. This is the substance of testimony from bank officials in Chicago, Ill., South Bend, Ind., Richmond, Ind., Vincennes, Ind., and Terre Haute, Ind.

Testimony to the effect that the United States Rule, as based on Chancellor Kent's decision, is used by the Federal Courts, is at hand from the Judge of the United States District Court for Indiana.

2. \$125. ELKHART, IND., Sept. 1, 1900.

Ten months after date, I promise to pay to the order of Silas Kulp, One Hundred Twenty-five and $\frac{no}{100}$ Dollars, value received, with interest at 7%. ARTHUR SWAZEE.

Indorsements: Nov. 1, 1900, \$40; Dec. 24, 1900, \$50; March 1, 1901, \$25. What was due at settlement? (Merchant's Rule.)

3. \$750. IRELAND, IND., Feb. 26, 1900.

Ninety days after date, I promise to pay Burk & Sims, or order, Seven Hundred Fifty and $\frac{no}{100}$ Dollars, with interest at 6%, value received. GEO. HENTY.

Payments: Aug. 1, 1900, \$75; Dec. 28, 1900, \$110; March 1, 1901, \$145; July 1, 1901, \$225. What was due Dec. 24, 1901? (Merchant's Rule.)

4. \$534.19. KNOXVILLE, IND., May 1, 1900.

Nine months after date, I promise to pay Henry Smythe at Knoxville Bank, Five Hundred Thirty-four and $\frac{19}{100}$ Dollars, with interest at 6%. STREETOR IVES.

Payments: Aug. 15, 1900, \$300; Oct. 1, 1900, \$75; Dec. 1, 1900, \$80. What was due Feb. 1. 1901? (Merchant's Rule.)

By the custom of some banks and some business men the interest computations are based upon the actual number of days rather than the calendar months.

The third and fourth above should be solved by finding the actual number of days in the time periods.

COMMERCIAL PAPER

Commercial paper embraces notes, checks, drafts, bills of exchange, letters of credit, etc.

A promissory note (called a note) is a written promise by one party to pay another a specified sum of money at a specified time.

The *maker* of a note is the one who promises to pay the sum specified at the specified time.

The *payee* of a note is the person to whom the promise is made. The promise may be made to his order.

The *face* of a note is the sum specified in the note, *not* including any interest.

The following is a form of promissory note:

$400. VINCENNES, IND., Feb. 4, 1902.

One year after date, I promise to pay to Henry Hazen, or order, Four Hundred and $\frac{no}{100}$ Dollars, value received, with interest at 5%. JOHN SIKES.

John Sikes is the *maker*, Henry Hazen the *payee*, and $400 is the *face*.

This note is *negotiable*, or *transferable*. Any note made payable to the *bearer* or to the *order* of the payee is negotiable. The above note can be transferred by the payee writing his name across the back of the note. This act is called *indorsing* a note, and when so indorsed the note is payable to any one who is the lawful holder of it. A note payable to *bearer* does not need indorsement. An indorsement may specify a particular person to whom the payment must be made, and when so indorsed is not further transferable. Thus, Henry Hazen may write across the back of the note:

Pay to Ezra Markle,

HENRY HAZEN.

If this should read

Pay to Ezra Markle, or order,
HENRY HAZEN,

the note would be still negotiable.

In any instance the *indorser* of a note becomes responsible for its payment, unless the indorsement reads,

Pay to Ezra Markle, without recourse,
HENRY HAZEN.

It is becoming quite common to specify a bank as the place at which the note is payable, and when so specified the note should be presented at the bank when due.

DAYS OF GRACE

Formerly it was a general custom to allow three days, called "days of grace," for the payment of a note after it was nominally due. This custom is supposed to have originated in ancient times between the Jews and Gentiles. Since Saturday was a Jewish and Sunday a Gentile holiday, and since these two are *consecutive* days, it was decided at court to allow *three days* in which to make payment of a note after it was due, without the process of the law. Thus, a note drawn up for 60 days could not be collected until the 63d day after date.

The following states have abolished the "days of grace" by statute: California, Colorado, Connecticut, District of Columbia, Florida, Idaho, Illinois, Maine, Maryland, Massachusetts, Montana, North Dakota, New Hampshire, New Jersey, New York, Ohio, Oregon, Pennsylvania, Utah, Vermont, and Wisconsin. In other states three days are allowed unless it is specified in the note "without grace."

TABLE SHOWING THE NUMBER OF DAYS FROM ANY DAY OF ANY MONTH TO THE SAME DAY OF ANY MONTH NOT MORE THAN ONE YEAR LATER

From	To Jan.	Feb.	Mch.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Jan.	365	31	59	90	120	151	181	212	243	273	304	334
Feb.	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	30	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
Aug.	153	184	212	243	273	304	334	365	31	61	92	122
Sept.	122	153	181	212	242	273	303	334	365	30	61	91
Oct.	92	123	151	182	212	243	273	304	335	365	31	61
Nov.	61	92	120	151	181	212	242	273	304	334	365	30
Dec.	31	62	90	121	151	182	212	243	274	304	335	365

NOTES. 1. If the interval of time under consideration includes the month of February of a *leap year*, add one day to the above.

2. If the period of time be over one year, to the number of days obtained from the table add 365 days as many times as there are years.

3. Examples. The number of days from April 14 to Sept. 23 = 153 days + 9 days = 162 days. The number from Oct. 26 to May 11 = 212 days − 15 days = 197 days.

4. This table is *not* helpful in finding *simple interest*, but for *accurate* interest and *bank discount* it is very convenient.

BANKS AND BANKING

The word "bank" is from *banco*, meaning a bench or table, on which the Venetian money changers displayed their money. If any one of them failed, the creditors broke the bench in pieces; thus, the money changer became a *bankrupt*.

Banks are very old institutions. In the Metropolitan Museum of Art in New York, there are tablets containing inscriptions which prove that the Babylonians, under the reign of Nebuchadnezzar, did a kind of banking business as early as 600 B.C.

In Luke xix. 23, is found the word "bank." "Wherefore then gavest not thou my money into the *bank*, that at my coming I might have required mine own with usury?"

It is known that the Chinese, the Greeks, and the Romans had banks which exercised in the main the functions of a modern bank, except that they did not issue notes.

Among the earlier banks of Europe may be named the Bank of Venice, 1171; of Barcelona, 1401; of Genoa, 1407; and of Amsterdam, 1609. Many of the ideas of modern banking are traceable to those of the Bank of Amsterdam.

The Bank of England, one of the greatest institutions of its kind in the world, was founded in 1694; its principal projector being one William Patterson, who visited the bank at Amsterdam and the banks among the colo-

nists of Massachusetts for ideas which he incorporated in the Bank of England. Its capital in the beginning was £1,200,000 ($6,000,000), which has been increased to £15,000,000 ($75,000,000) at present.

The business of banking was carried on by the colonists of America as early as 1753.

The first bank of the United States was incorporated by Act of Congress in 1791 with a capital of $10,000,000, being chartered for 20 years. It was rechartered in 1816 for 20 years with a capital of $35,000,000. President Jackson vetoed a bill to recharter it in 1836.

Our national banking system is an outgrowth of the management of the finances of the United States at the close of the Civil War. Besides the national banks of the United States there are state and private banks and loan and trust companies. By a recent report there were 4426 national banks, 4191 state banks, 942 savings banks, 756 private banks, and 260 loan and trust companies. By the same report the individual deposits subject to check were: in national banks, $3,111,690,195; in state banks, $1,164,020,972; in private banks, $64,974,392; in the vaults of loan and trust companies, $835,499,064; and in savings banks, $2,179,468,299; or a total deposit of $7,355,652,922 subject to individual check.

THE BUSINESS OF A BANK

1. At first a bank was simply a place where money was received for safe keeping, the same being paid out on the presentation of checks properly signed.

2. Since a certain portion of the money thus received was available for use, the banks gradually began the loaning of this money on promissory notes, keeping enough on hand to pay the expected demands by checks.

3. The money received on deposit might be used to discount notes "which individuals might own and upon which the holders might wish to realize money before the notes were due."

NOTE. The charge made by a bank for advancing money on notes before they are due is called *a discount*. *Discount* is any interest paid in advance.

4. Banks sell and collect drafts on other banks or corporations, charging a fee for the same.

NOTE. A bank draft is a written order from one bank to another directing it to pay a third party a certain sum of money at a specified time.

5. The national banks of the United States usually issue promissory notes which circulate as money. For the privilege of issuing these notes the bank is required to deposit government bonds with the Treasurer of the United States in a larger sum than that of the notes issued.

Except in savings banks, money placed on deposit does not usually draw interest, being deposited for safe keeping simply.

The following is the usual form of check used to present for drawing out a deposit or a part of it.

No. .... *Evansville, Ind.,* ....

The First National Bank of Evansville

Pay to ...., *or order,*

....*Dollars,*

$ ....

Checks may be made payable to *bearer*, to the *order* of *the payee*, or to *self*.

If an individual wishes to obtain money from a bank in which he has *no* deposit, he may secure it by giving his individual note, or by selling to the bank a note on another individual. In either case he must give security which is satisfactory to the bank officials. If an individual sells to a bank a note not yet due, the bank discounts it for the time from the date of the discount to the date of maturity.

There are two kinds of notes sold to banks:

1. Those due at a future date without interest.
2. Those due at a future date with interest.

Notes are usually discounted at banks for short periods, as for 30 da., 90 da., 3 mo., 4 mo., "days of grace" being added except in states where they are abolished.

In maturing a note drawn in days, it is customary to count forward by days and add the days of grace if allowable. Thus, a note dated July 14 at 60 da. matures 63 da. after July 14, or Sept. 15; 17 more days in July, 31 in August, and 15 in September make up the 63 da. In maturing a note drawn in months, it is customary to count forward by *months*. Thus, a note dated July 14 at 2 mo. matures 2 mo. and 3 da. after July 14, or Sept. 17.

The term of discount, or "time to run," is the time from the *date of discount* to the *date of maturity*, and is always found by *counting the actual number* of days. Thus, a note dated June 3 at 90 da. and discounted July 25 matures Sept. 4. From July 25 to Sept. 4 is 41 da. Most banks use 360 da. to the year. In the above instance the bank would charge interest for $\frac{41}{360}$ of a year.

Hence, to find the proceeds of a note without interest:

I. Find the date of *maturity*.

II. Find the *term of discount* in days.

III. Find the interest on the face of the note for the term of discount, and call it the *bank discount*.

IV. Subtract the bank discount from the face of the note, and call the result the *proceeds*.

To find the proceeds of a note bearing interest: Find the amount of the note at simple interest from date to maturity, then discount this sum as is indicated above.

Banks always get more than legal interest by their way of discounting notes; but usage allows this. An extreme case will illustrate: Discount $600 at bank for 10 yr. at 6%. The bank discount is $360. The proceeds is $240. This sum on interest for 10 yr. at 6% draws but $144. Thus the bank gets $216 more than legal interest.

The custom of the banks of discounting notes for short periods is a good one, for a large portion of the capital invested in loans is from the individual deposits, which are subject to "call" on short notice, and the banks own circulation which must be redeemed on presentation.

In case of a heavy demand for redemption of bills issued or of deposits on bond, the bank relies upon the early maturity of notes due. There is risk in long-time loans from the possibility of fluctuations in securities.

EXERCISES

1. A note of $280 dated Jan. 24, 1901, at 90 da., was discounted at bank March 1, 1901, at 8%. Find the date of maturity, the term of discount, the bank discount, and the proceeds.

SOLUTION. 93 da. after Jan. 24 is April 27, which is the date of maturity; from March 1 to April 27 is 57 da., which is the term of discount; the interest on $280 at 8% for 57 da. is $3.55, which is the bank discount; $280 − $3.55 = $276.45, which is the proceeds.

This means that if one should give his note at bank for $280 for 90 da., the bank would keep $3.55 for advancing the maker of the note $276.45, which he would have the use of for 93 da.

2. A note of $600 dated July 24, at 3 mo., was discounted at bank Sept. 1, at 8%. Find the proceeds.

SOLUTION. This note matures 3 mo. 3 da. after July 24, or Oct. 27. The term of discount is from Sept. 1 to Oct. 27 = 56 da. The bank discount on $600 for 56 da. at 8% = $7.47. Therefore, the proceeds = $600 − $7.47 = $592.53.

3. A note of $450 dated April 1, at 2 mo., bearing interest at 6%, was discounted at bank April 28, at 8%. Find the proceeds.

SOLUTION. The simple interest on $450 for 2 mo. 3 da. at 6% = $4.725; hence the amount discounted is $454.725.

The date of maturity is June 4; the term of discount is 37 da.; the bank discount on $454.725 at 8% for 37 da. = $3.739; the proceeds = $454.725 − $3.739 = $450.986.

If this note had been drawn at 60 da. instead of 2 mo., the date would have been June 3; hence, the proceeds would have been larger.

Find the date of maturity, the term of discount, the bank discount, and the proceeds of the following (with grace):

	Face	Date	Time drawn	Date of Disc.	Rate of Disc.
4.	$240	Apr. 15	60 da.	Apr. 15	8%
5.	$750	Mar. 5	90 da.	Apr. 9	7%
6.	$406	June 19	3 mo.	Aug. 4	8%
7.	$600	Oct. 7	2 mo.	Nov. 10	6%
8.	$75.60	Jan. 1	30 da.	Jan. 16	8%
9.	$1000	Dec. 20	4 mo.	Feb. 1	7%
10.	$60.19	Aug. 4	60 da.	Sept. 6	8%
11.	$500	Sept. 1	3 mo.	Nov. 1	7%

The following are interest-bearing notes:

	Face	Rate of Int.	Date	Drawn for	Date of Disc.	Rate of Disc.
12.	$300	6%	May 7	90 da.	July 1	8%
13.	$175.50	5%	June 19	2 mo.	July 1	7%
14.	$219.75	6%	Feb. 17	60 da.	Mar. 10	7%
15.	$817.25	5%	Mar. 15	30 da.	June 1	6%

16. A note of $750 without interest, dated Feb. 14, at 90 days, was discounted 60 days before maturity at bank at 8%. Find the proceeds.

17. A note of $430 without interest, dated July 14, at 3 months, was discounted 1 mo. 3 da. after date at bank at 7%. Find the proceeds.

18. $700. CINCINNATI, OHIO, Dec. 14, 1900.

Three months after date I promise to pay Julius Job, or order, seven hundred and $\frac{no}{100}$ dollars for value received.

ZENAS CASE.

Mr. Job wishes to discount this note at bank Jan. 1, 1901. How much will he realize?

If Mr. Job had sold this note to Silas Breen, instead of to the bank, on Jan. 1, 1901, how would he have indorsed it so that Mr. Breen alone could have received the money?

19. For how much must I give my note at bank, due in 60 days, to realize $350, if discounted at 7%?

NOTE. Discount $1 for the time at 7% and divide $350 by the result.

20. For how much must I give my note at bank, due in 3 months, to realize $400, discounted at $7\frac{1}{2}$%?

21. A note of $500 was discounted 40 days before maturity, and the proceeds were $495.60. What was the rate of discount?

22. $560. CINCINNATI, OHIO, Jan. 25, 1900.

Sixty days after date, I promise to pay William Slonaker, or order, Five Hundred Sixty and $\frac{no}{100}$ Dollars, with interest at 6%.

JAMES REMER.

What were the proceeds if discounted Feb. 1, at 7% ?

23. What does a bank make on $75,000 if borrowed for 1 year at 4%, and discounts it all at 8% on an average, every 60 days? How many days in the last term of discount if the year be 1903?

24. Write a check for $175.45, payable to yourself.

25. Write a check payable to your order, and indorse it so that James Henry alone can collect it.

26. Write a check payable to your order, and indorse it so that James Henry, or his order, can collect it.

27. How would you indorse the last one above so that you would not be responsible for it after transferred?

28. "I owe $6000, and to pay it I had a note of $3500, maturing in 18 days, discounted at bank at 8%, and drew a second note, payable in 2 months, for such a sum that when discounted at the same rate the proceeds of both notes enabled me to pay the debt. What was the face of the second note?

Ans. $2549.70."

INTEREST TABLES

Banking houses and trust companies, where the bulk of interest computations for the country are made, frequently use books containing interest tables. From a considerable amount of testimony it is learned that interest tables are used for convenience in checking only — no time being gained, since there are so many short methods for calculating interest at 360 days to the year. Several of the large banking houses in Chicago make their own calculations for each interest computation, and use tables occasionally to check results. Interest tables

are not "regarded as conclusively correct" by the United States District Court for the state of Indiana.

CONVENIENT RULES FOR COMPUTING INTEREST

One day's interest at 36% is found by moving the decimal point 3 places to the left in any principal. Thus, 1 day's interest at 36% on $745 is $0.745. Three days' interest at 12% is found by moving the decimal point 3 places to the left in any principal. The reason is very plain. Six days' interest at 6% is found by moving the decimal point 3 places to the left in any principal. For 60 days move the decimal point 2 places to the left. Whence, for 4% take $\frac{1}{3}$ of the result at 12%. For 5% take $\frac{5}{6}$ of the result at 6%. For 8% take $\frac{2}{3}$ of the result at 12%, or $\frac{4}{3}$ of the result at 6%. Then change any of these results to suit the time. For example:

Find the interest on $465 at 7% for 30 days.

$$\$0.465 = \text{the int. for 6 days at } 6\%.$$
$$\tfrac{7}{6} \text{ of } \$0.465 = \$0.54\tfrac{1}{4} = \text{int. for 6 days at } 7\%.$$
$$\$0.54\tfrac{1}{4} \times 5 = \$2.71\tfrac{1}{4} = \text{int. for 30 days at } 7\%.$$

Or this:

$$\$0.465 = \text{the int. for 6 days at } 6\%.$$
$$\$0.465 \times 5 = \$2.325 = \text{int. for 30 days at } 6\%.$$
$$\tfrac{7}{6} \text{ of } \$2.325 = \$2.71\tfrac{1}{4} = \text{int. for 30 days at } 7\%.$$

A set of interest tables sufficiently complete to be of much assistance in interest computations would make a good-sized volume. Hence, this is impracticable in this work. However, three tables are made. Table I shows the interest on the sums designated at the top of the page at 6% for any number of days from 1 to 29 inclusive, accurate to the nearest mill.

Table II shows the same kind of results for 1 to 11 months inclusive.

Table III shows the same for 1, 2, 3, 4, or 5 years. With some additions, Table III may be more extended.

INTEREST TABLE I

INTEREST AT 6%

Da.	$100	$200	$300	$400	$500	$600	$700	$800	$900	$1000	Da.
1	0.017	0.033	0.050	0.067	0.083	0.100	0.117	0.133	0.150	0.167	1
2	0.033	0.067	0.100	0.133	0.167	0.200	0.233	0.267	0.300	0.333	2
3	0.050	0.100	0.150	0.200	0.250	0.300	0.350	0.400	0.450	0.500	3
4	0.067	0.133	0.200	0.267	0.333	0.400	0.467	0.533	0.600	0.667	4
5	0.083	0.167	0.250	0.333	0.417	0.500	0.583	0.667	0.750	0.833	5
6	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000	6
7	0.117	0.233	0.350	0.467	0.583	0.700	0.817	0.933	1.050	1.167	7
8	0.133	0.267	0.400	0.533	0.667	0.800	0.933	1.067	1.200	1.333	8
9	0.150	0.300	0.450	0.600	0.750	0.900	1.050	1.200	1.350	1.500	9
10	0.167	0.333	0.500	0.667	0.833	1.000	1.167	1.333	1.500	1.667	10
11	0.183	0.367	0.550	0.733	0.917	1.100	1.283	1.467	1.650	1.833	11
12	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000	12
13	0.217	0.433	0.650	0.867	1.083	1.300	1.517	1.733	1.950	2.167	13
14	0.233	0.467	0.700	0.933	1.167	1.400	1.633	1.867	2.100	2.333	14
15	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000	2.250	2.500	15
16	0.267	0.533	0.800	1.067	1.333	1.600	1.867	2.133	2.400	2.667	16
17	0.283	0.567	0.850	1.133	1.417	1.700	1.983	2.267	2.550	2.833	17
18	0.300	0.600	0.900	1.200	1.500	1.800	2.100	2.400	2.700	3.000	18
19	0.317	0.633	0.950	1.267	1.583	1.900	2.217	2.533	2.850	3.167	19
20	0.333	0.667	1.000	1.333	1.667	2.000	2.333	2.667	3.000	3.333	20
21	0.350	0.700	1.050	1.400	1.750	2.100	2.450	2.800	3.150	3.500	21
22	0.367	0.733	1.100	1.467	1.833	2.200	2.567	2.933	3.300	3.667	22
23	0.383	0.767	1.150	1.533	1.917	2.300	2.683	3.067	3.450	3.833	23
24	0.400	0.800	1.200	1.600	2.000	2.400	2.800	3.200	3.600	4.000	24
25	0.417	0.833	1.250	1.667	2.083	2.500	2.917	3.333	3.750	4.167	25
26	0.433	0.867	1.300	1.733	2.167	2.600	3.033	3.467	3.900	4.333	26
27	0.450	0.900	1.350	1.800	2.250	2.700	3.150	3.600	4.050	4.500	27
28	0.467	0.933	1.400	1.867	2.333	2.800	3.267	3.733	4.200	4.667	28
29	0.483	0.967	1.450	1.933	2.417	2.900	3.383	3.867	4.350	4.833	29

INTEREST TABLE — II

INTEREST AT 6%

Mo.	\$100	\$200	\$300	\$400	\$500	\$600	\$700	\$800	\$900	\$1000	Mo.
1	0.500	1.000	1.500	2.000	2.500	3.000	3.500	4.000	4.500	5.000	1
2	1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000	2
3	1.500	3.000	4.500	6.000	7.500	9.000	10.500	12.000	13.500	15.000	3
4	2.000	4.000	6.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	4
5	2.500	5.000	7.500	10.000	12.500	15.000	17.500	20.000	22.500	25.000	5
6	3.000	6.000	9.000	12.000	15.000	18.000	21.000	24.000	27.000	30.000	6
7	3.500	7.000	10.500	14.000	17.500	21.000	24.500	28.000	31.500	35.000	7
8	4.000	8.000	12.000	16.000	20.000	24.000	28.000	32.000	36.000	40.000	8
9	4.500	9.000	13.500	18.000	22.500	27.000	31.500	36.000	40.500	45.000	9
10	5.000	10.000	15.000	20.000	25.000	30.000	35.000	40.000	45.000	50.000	10
11	5.500	11.000	16.500	22.000	27.500	33.000	38.500	44.000	49.500	55.000	11

INTEREST AT 6% — III

Yr.	\$100	\$200	\$300	\$400	\$500	\$600	\$700	\$800	\$900	\$1000	Yr.
1	6.00	12.00	18.00	24.00	30.00	36.00	42.00	48.00	54.00	60.00	1
2	12.00	24.00	36.00	48.00	60.00	72.00	84.00	96.00	108.00	120.00	2
3	18.00	36.00	54.00	72.00	90.00	108.00	126.00	144.00	162.00	180.00	3
4	24.00	48.00	72.00	96.00	120.00	144.00	168.00	192.00	216.00	240.00	4
5	30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	270.00	300.00	5

By use of the tables find the interest on \$435 for 2 yr. 4 mo. 17 da. at 5%.

Interest on \$400 for 2 yr. at 6% . .	= \$48.00
Interest on \$400 for 4 mo. at 6% . .	= 8.00
Interest on \$400 for 17 da. at 6% . .	= 1.133
Interest on \$30 for 2 yr. at 6% . .	= 3.60
Interest on \$30 for 4 mo. at 6% . .	= .60
Interest on \$30 for 17 da. at 6% . .	= .085
Interest on \$5 for 2 yr. at 6% . .	= .60
Interest on \$5 for 4 mo. at 6% . .	= .10
Interest on \$5 for 17 da. at 6% . .	= .014

∴ Interest on \$435 for 2 yr. 4 mo. 17 da. at 6% = \$62.132. $\frac{5}{6}$ of \$62.132 = \$51.777. ∴ Int. at 5% = \$51.78.

Without the use of tables the following exhibits a plan:

$$
\begin{array}{rl}
\$435 & \\
.05 & \\
\hline
\$21.75 & = \text{int. for 1 yr.} \\
2 & \\
\hline
\$43.50 & = \text{int. for 2 yr.} \\
7.25 & = \text{int. for 4 mo.} \\
.725 & = \text{int. for 12 da.} \\
.241 & = \text{int. for 4 da.} \\
.060 & = \text{int. for 1 da.} \\
\hline
\$51.776 & = \text{int. for 2 yr. 4 mo. 17 da.}
\end{array}
$$

The latter solution is much shorter and easier, and is therefore less liable to error than the former, and will always be so, unless the set of tables is very much more extended.

Since banks do not deal with "long time" notes, except in their collection departments, it is apparent that the tables are useful in finding bank discount. For example, a note of \$300 is discounted at bank for 63 days at 7%.

By reference to the tables the interest on \$300 for 60 days at 6% = \$3, and the interest for 3 days = \$0.15. ∴ The bank discount on \$300 for 63 days at 6% = \$3.15, and at 7% = $\frac{7}{6}$ of \$3.15 = \$3.675.

EXCHANGE

If Mr. Ream of Terre Haute owes Mr. Jekyl of Indianapolis \$300, he could go in person and pay the amount directly to Mr. Jekyl. This would be a costly way to do

it. Mr. Ream could put the $300 into an envelope and send it to Mr. Jekyl at an expense of a postage stamp. This method would be at the risk of losing it all.

He could put it into an envelope and have it registered by the Post-office Department at a cost of 10 cents, whereby the risk would be much less.

He could send it in a package by express by paying the Express Company a small fee for transmitting it.

In each of the four methods of paying the debt mentioned above the money is actually transmitted to Mr. Jekyl.

Mr. Ream may know that Mr. Jekyl of Indianapolis owes a Mr. Jasper of Terre Haute the same sum, — $300.

If this is the case Mr. Ream could pay Mr. Jasper the $300 and cancel his debt to Mr. Jekyl, and Mr. Jekyl's debt to Mr. Jasper, and send no money to or from Indianapolis. But this is impracticable.

Mr. Ream could go to the post office, the telegraph office, or the express office, in Terre Haute, and buy an order for the amount for a small fee extra, and send the order to Mr. Jekyl, who in turn could go to the proper office in Indianapolis and draw the $300.

He could send his personal check on a home bank for $300, and Mr. Jekyl could present the check at a bank and probably draw his money; at least, he could by paying a small fee called *exchange*.

Or, he could buy a draft from a home bank, on a bank in another city, say Chicago, and send it to Mr. Jekyl, who would present it to a bank at Indianapolis and draw his money, and the debt would thus be paid. Thus, it is seen that there are many ways of paying the debt of $300 *without* actually sending the money.

The bank to which the draft is directed is called a *correspondence bank*, and is usually in a larger city than the *issuing bank*.

Terre Haute purchases more in Chicago than Chicago does in Terre Haute. Terre Haute owes Chicago more than Chicago owes Terre Haute. Hence, at any time that these cities wish to make settlement, Terre Haute must send the money for the balance to Chicago. This incurs some expense. The banks in Terre Haute, therefore, charge a small fee, called *exchange*, to meet this expense.

Since the banks at Chicago do not have to send any money to the banks at Terre Haute, they are at no expense of this kind.

Hence, a draft issued by a bank at Chicago on a bank at Terre Haute could be bought at *par*, and sometimes *below par*, or at a *discount*, because the Chicago banks want money and not drafts.

A draft may be purchased at a *premium*, or above par, at par, or at a discount when it is *below par*. Drafts drawn up in smaller cities on banks in larger cities are usually at a premium, because the large city is demanding that the balance in her favor be paid in money. The smaller city must be at expense to do this, or pay interest on the balance.

Exchange is a method of paying a debt between two places without actually sending the money, but by the use of drafts or bills of exchange.

If the issuing bank issues a time draft, at 60 days say, it does not have to make payment for 63 days to settle the balance. Since it keeps the money paid in for the draft for the specified time, it allows some interest, usually at a specified rate per cent for the time. (Days of grace are subject to the same usage as in bank discount.)

FORM OF A DRAFT

$240. TERRE HAUTE, IND., Dec. 8, 1901.

At sight, pay to the order of Jenkins & Co., Two Hundred Forty and $\frac{no}{100}$ Dollars, value received, and charge to the account of

EXCHANGE NATIONAL BANK.

To Ninth National Bank,
N.Y.

A time draft differs from the above in that instead of "at sight" the time is mentioned.

EXERCISES

1. Find the cost of a sight draft for $300, premium at rate of $1\frac{1}{2}\%$.

2. Find the cost of a sight draft for $419 at a discount of $\frac{3}{4}\%$.

3. Find the cost of a 60-day draft for $450, interest at 6%, and at a premium of 1%.

SOLUTION. The interest on $450 at 6% for 63 days = $4.725. The premium on $450 at 1% = $4.50. The cost = $450 + $4.50 – $4.725 = $449.775.

4. Find the cost of a draft for $750 at 30 days, discount $1\frac{1}{4}\%$, and interest at 6%.

SOLUTION. The interest on $750 at 6% for 33 days = $4.125. The discount on $750 at $1\frac{1}{4}\%$ = $9.375. ∴ The cost = $750 – $9.375 – $4.125 = $736.50.

5. Find the cost of a sight draft for $680, premium = $\frac{7}{8}\%$.

6. Find the cost of an 80-day draft for $960, premium = $\frac{5}{8}\%$, and interest = 5%.

7. Find the cost of a 60-day draft for $4700, premium $\frac{4}{5}\%$, interest 5%.

8. Find the face of a draft which cost $1200, premium $1\frac{1}{4}\%$.

9. Find the face of a draft which cost $900, discount $\frac{3}{5}\%$.

10. Find the face of a 60-day draft which cost $975, discount $1\frac{1}{2}\%$, interest 6%, no grace.

11. If a sight draft for $2500 on Boston can be bought in San Francisco for $2475 in exchange at a premium or discount, what is the rate of exchange?

12. Find the cost of a draft for $750 payable in 60 days after sight, exchange being $\frac{1}{2}\%$ premium, interest 7%.

13. Find the face of a draft on New York, at 90 days' sight, bought for $450, exchange $1\frac{3}{4}\%$ premium, interest 5%.

14. What is the face of a draft on St. Paul for 60 days which may be bought for $1000, exchange being $\frac{7}{8}\%$ discount, interest 7%?

TIME DRAFT

$1600. ST. PAUL, MINN., Nov. 1, 1901.

Sixty days after date, pay to the order of Henry R. Brant, Sixteen Hundred and $\frac{no}{100}$ Dollars, and charge the same to the account of

BARTON SMITH.

To the Merchants' National Bank,
Chicago, Ill.

15. A merchant owes $2000 and has a 60-day draft for $1200 discounted at bank at 6%. If the rate of exchange is $1\frac{1}{2}\%$ premium, how much more money will he need to discharge the debt?

A *circular letter of credit* is a letter issued by a banking house to a person who wishes to travel abroad. If a citizen of St. Louis wishes to travel in Europe, he may take out a circular letter of credit from some banking house in St. Louis, or New York, addressed to bankers in the principal cities to be visited, asking them to furnish the holder with money within the limit of the letter. Whatever sum is advanced by the foreign banks is indorsed on the letter. The letter, when once issued, is not transferable.

EXERCISES

1. What is the cost of a bill of exchange on London for £480 *5s. 6d.*, exchange being $4.87 ?

2. What is the cost of 1780 marks, exchange $0.238, brokerage $\frac{1}{8}$% ?

3. What is the face of a bill of exchange on Liverpool for which $480 was paid, exchange $4.86 ?

4. How large a bill of exchange on Berlin can be bought for $1280, exchange at 96$\frac{7}{8}$?

5. A gentleman purchased a circular letter of credit from Drexel, Morgan & Co. At Paris he drew 1200 francs; at London he drew £120; at Berlin 700 marks. Exchange in Paris was $\frac{1}{2}$% premium; in London 1% premium; in Berlin $\frac{3}{4}$% premium. What did the letter cost him, if gold was worth $1.10 ?

STOCKS AND BONDS

A number of persons wish to organize a natural gas company for the purpose of furnishing gas to the people of Delphi, Ind. It is to be a business venture for the profit that may revert to the company.

After studying the situation very carefully it is decided to proceed, and the company is to be known as the Delphi Natural Gas Co. of Delphi, Ind. The capital of the company shall be $300,000, consisting of 6000 shares of $50 each.

The members who organize the company agree to take 3500 shares, par value $175,000, of the stock, thus securing a controlling interest. Then 2500 shares, par value $125,000, are put upon the market to be sold to outside persons, who are to share in the losses and gains of the company in proportion to the stock they hold. When this amount is all subscribed all subscriptions are to be binding.

It is arranged that the stock subscribed shall be payable in four equal installments. The company calls for the first installment ($12.50 per share) when the work begins. As the work of sinking wells, laying of pipes, etc., proceeds other installments are called for, until all of the installments are paid in.

The plant is finally completed and ready to furnish gas to customers.

Enough gas is sold at the end of three months to declare a *quarterly* dividend of 5% upon all of the stock of the company. Many who subscribed at first very reluctantly are now glad they did so. At the end of the next three months a second quarterly dividend of 5% is declared to be paid to the holders of stock.

By this time many persons who did not subscribe for stock are anxious to possess some of it, for a quarterly dividend of 5% is much more than most investments yield; hence, they are willing to give more than par value for the stock; probably they are willing to give $80 per share, par value $50, for that would command $12\frac{1}{2}$% on the investment. (Dividends are always declared on the par value of the stock.) Since $80 is 160% of $50, when the stock was selling at $80 per share, the quotation was 160%.

If, after a while, gas seems to be giving out, the stock may be purchased below par, say for $30 per share, par value $50. When the $50 shares are sold for $30 each, the quotation is 60%, since $30 is 60% of $50. Stock quoted at 160 is said to be at 60% *premium;* and when quoted at 60 it is said to be at 40% *discount.*

A stock company is a company formed for the transaction of business as an individual. If a company is chartered by the state, it becomes an incorporated company, or a corporation. It is then entitled to all privileges and is under all obligations due by law.

The *capital* of a company is the stock of the company, and is divided into *shares* of 10¢, $1, $5, $10, $50, or $100 each. Usually shares are $100 each, sometimes more. Those who hold one or more shares are called *stockholders.*

A *stock certificate* is a document setting forth the name

of the company, the amount of its capital, the size of a share, the number of shares of the company, and the number of shares held by the person named in the certificate.

The *par value* of a share is the value named on the face of it. The *market value* may be at par, above par, or below par.

An *installment* is a part of the capital.

An *assessment* is a sum levied by the company to meet reverses or to make repairs.

A *dividend* is a part or all of the net earnings of the company, and is usually declared to be a certain per cent of the par value, and is divided among the stockholders.

A *scrip* dividend is one which is paid in more stock in lieu of cash.

Preferred stock is frequently issued and is made to receive a *stated* dividend from the earnings of the company before any dividend is declared on *ordinary* or *common* stock. This stock is made *cumulative* sometimes, so that if in any year no dividend is paid to the holders of it, the earnings of any future year may be made to pay *back* dividends up to the "stated" amount. Because of more certainty of dividends, this preferred stock usually sells higher in the markets than the common stock.

A Bear on the stock market is one who is *short* in stock which he has promised for future delivery, and must buy to fill his promises. He therefore wishes to get stock as low as possible.

A Bull is one who has stock to sell and is *long* in stock. He thinks prices will advance, and buys to sell at this advanced price. *Hypothecating* stock is depositing it as security for money borrowed.

Watering stock consists of increasing the number of

shares without a corresponding increase in the capital of the company.

A *corner* is a condition in the stock market where an individual or firm controls a certain product, and thereby controls the price of it.

FORM OF A STOCK CERTIFICATE

No. 217   50 *Shares.*

St. Louis, Mo., 2/20, '02

This certifies that Joseph James

is entitled to 50 *shares of the capital stock of the Delphi Natural Gas Co. at Delphi, Ind.*

Frank McGee, *Sec'y.*   Fred'k Ward, *Pres.*

6000 shares . . $50 each.

The following tables show the maximum prices paid for the various stocks named on the New York Exchange:

Name	March 17, 1902	May 10, 1902
American Sugar	$128\frac{3}{4}$	128
American Sugar p'f'd	118	119
American Locomotive	32	$32\frac{1}{4}$
American Locomotive p'f'd	93	$94\frac{1}{2}$
American Linseed Oil	$25\frac{3}{8}$	26
American Linseed Oil p'f'd	58	54
Baltimore & Ohio	$106\frac{3}{8}$	$107\frac{1}{2}$
Chicago & Alton	$36\frac{1}{8}$	$36\frac{1}{2}$
C., M. & St. P.	165	$170\frac{1}{8}$
C. &. N. W.	252	253
C., R. I. & P.	173	175
Chesapeake & Ohio	$46\frac{3}{8}$	$47\frac{3}{4}$

Name	March 17, 1902	May 10, 1902
Chicago & Great Western	$24\frac{1}{2}$	30
Chicago & Great Western p'f'd	$87\frac{3}{4}$	89
Denver & Rio Grande	44	42
Erie	$36\frac{3}{4}$	37
Erie 1st p'f'd	$68\frac{1}{4}$	68
Erie 2d p'f'd	$54\frac{1}{2}$	$53\frac{1}{8}$
Illinois Central	$140\frac{1}{2}$	$152\frac{7}{8}$
Louisville & Nashville	$104\frac{3}{8}$	$142\frac{1}{2}$
Missouri Pacific	$101\frac{3}{8}$	$100\frac{3}{8}$
National Biscuit	$51\frac{3}{8}$	$49\frac{1}{2}$
National Biscuit p'f'd	$108\frac{1}{2}$	$107\frac{1}{2}$
N. Y. Central	163	158
Pennsylvania Railroad	$151\frac{3}{8}$	$150\frac{1}{4}$
Southern Railway	$33\frac{1}{8}$	$37\frac{3}{8}$
Southern Railway p'f'd	$97\frac{1}{8}$	$95\frac{1}{4}$
Union Pacific	$99\frac{5}{8}$	$104\frac{1}{4}$
Union Pacific p'f'd	$87\frac{1}{8}$	$87\frac{3}{4}$
United States Steel	$42\frac{7}{8}$	$41\frac{3}{8}$
United States Steel p'f'd	$95\frac{3}{4}$	$91\frac{7}{8}$
Wabash	$23\frac{7}{8}$	$26\frac{3}{4}$
Wabash p'f'd	$43\frac{3}{4}$	$44\frac{3}{4}$
Western Union	$91\frac{3}{8}$	$92\frac{3}{8}$

Government Bonds

Coupon 4's	112	$111\frac{1}{4}$
Registered 4's	111	$111\frac{1}{4}$
Coupon 4's, new	$139\frac{1}{4}$	$137\frac{1}{4}$
Registered 4's, new	139	$137\frac{1}{2}$
Coupon 5's	106	$105\frac{3}{4}$
Registered 5's	106	$105\frac{3}{4}$

Closing Curb Prices

Standard Oil	635 bid	640 bid
Elgin Watch	2050 bid	2200 bid
Chemical National Bank	4200 bid	4200 bid
Corn Exchange National Bank	450 bid	480 bid
Chase National Bank	700 bid	750 bid
Fifth Avenue National Bank	3500 bid	3550 bid
Manhattan National Bank	320 bid	340 bid

BONDS

If an incorporated company needs more money than its capital represents, it is customary for it to issue *bonds* which are sold for the money.

These bonds bear a specified rate of interest and are payable at a specified time. They are notes given by the Company and are secured by mortgages.

Conservative investors usually prefer the bonds to the stock of a company, for the interest is to be paid whether there is any dividend declared on stock or not.

The federal government, state, county, township, and city governments issue bonds to sell to secure money to pay off indebtedness, or to secure money to make needed improvements.

The advantage of issuing bonds is shown when a county wishes to build a court-house. Instead of laying a tax sufficiently heavy to pay for it at once these bonds are issued, payable in several annual installments, thus distributing the payment of the cost of the court-house over several years.

The expense of a war is too great to be paid off at once; hence, it is convenient to issue bonds in order to distribute the debt over a longer period. Any public financial burden can be thus apportioned over a series of years.

Bonds are bought and sold in the market in the same manner that stocks are.

Investors usually apply to *brokers* who are connected with a stock exchange, where stocks and bonds are sold. The broker charges a fee for his work which is called *brokerage*. The broker receives a double fee for each transaction, a fee from the seller and one from the buyer, for the same stock.

EXERCISES

1. What is the entire cost of 14 shares Penn. R.R. stock at 151?

SOLUTION. 151 means \$151 per share of \$100. ∴ the cost of 14 shares = \$151 × 14 = \$2114.

2. What is the entire cost of 300 shares of C. & E. I. stock at 168?

3. If 64 shares of E. & T. H. stock cost \$5120, what is the cost of one share? What is the quotation?

4. If stock is quoted at 96, how many shares will \$19200 buy?

5. State the three general problems illustrated in Exercises 2, 3, and 4.

6. What will 80 shares of B. & O. stock cost at $109\frac{3}{8}$, brokerage $\frac{1}{8}$?

SOLUTION. Since \$$109\frac{3}{8}$ is paid for each share, and the broker is paid \$$\frac{1}{8}$ for each share, the total cost of 1 share is \$$109\frac{3}{8}$ + \$$\frac{1}{8}$ = \$$109\frac{1}{2}$. Then the cost of 80 shares = \$$109\frac{1}{2}$ × 80 = \$8760.

7. Find the entire cost of 248 shares of Con. Gas Stock at $224\frac{3}{4}$, brokerage $\frac{1}{8}$.

8. What are the proceeds of 300 shares of C., R. I. & P. stock sold at $175\frac{1}{8}$, brokerage $\frac{1}{8}$?

SOLUTION. Since the broker charged \$$\frac{1}{8}$ per share, the net price of a share equals \$$175\frac{1}{8}$ − \$$\frac{1}{8}$ = \$175. The proceeds of 300 shares = \$175 × 300 = \$52500.

9. How much should I receive for 800 shares of C. & N. W. stock if sold at 252, which was the quotation April 5, 1902, brokerage $\frac{1}{8}$?

10. In Ex. 9 what was the total fee of the broker?

Remember the broker gets a fee from the buyer as well as from the seller.

11. If a broker receives \$16 for selling stock at the rate of $\frac{1}{8}\%$, how many \$100 shares did he sell?

SOLUTION. $\frac{1}{8}$ of \$100 = \$$\frac{1}{8}$. \$16 ÷ \$$\frac{1}{8}$ = 128. ∴ he sold 128 shares.

12. If a broker receives $7.50 for buying stock at the rate of $\frac{1}{8}\%$, how many $100 shares did he buy?

13. If a broker receives $8.50 for selling stock at the rate of $\frac{1}{8}\%$, how many $50 shares did he sell?

14. If a broker sold 256 shares of stock, $100 each, at $\frac{1}{8}$, what did he receive for selling?

15. If a broker bought 136 shares of stock, $50 each, what did he receive for buying?

16. If a broker sold 400 shares of stock, $100 each, and received $50 as brokerage, what was the rate per cent of brokerage?

17. If a broker bought 640 shares of stock, $25 each, and his fee for buying was $20, what was the rate per cent of brokerage?

18. State the three general problems illustrated in Exercises 11, 14, and 16. The same for Exercises 12, 15, and 17.

19. If the stock mentioned in Ex. 12 was quoted at $62\frac{1}{2}$, what was the total cost to the purchaser of the stock? What was the net amount received by the seller?

The broker did not buy for himself.

20. If a man purchased 125 shares of bank stock bearing 6% dividend annually, what was the annual income from his investment?

Solution. If the par value of a share was $100, the income from 1 share was $6, and from 125 shares, $6 × 125 = $750. Or this: the par value of 125 shares = $12,500. 6% of $12500 = $750. ∴ his income = $750.

21. If a man receives $800 annually from 5% stock, how many shares of $100 each does he possess?

Solution. 5% stock draws $5 per share of $100. $800 ÷ $5 = 160. ∴ he possesses 160 shares.

22. If a man receives $960 income annually from 160 shares of stock, what is his rate of income, or what kind of stock does he hold?

Solution. $960 ÷ 160 = $6. ∴ $6 per share is received. But $6 is 6% of $100. ∴ he holds 6% stock.

23. If I buy 50 shares of C. & E. I. 6% preferred stock, what is my annual income?

24. If I buy a sufficient amount of 5% preferred E. & T. H. stock to make an annual income of $480, how many shares did I buy?

25. If I buy 240 shares of I. C. R.R. stock and receive annually $1320, what kind of stock did I buy? That is, what is the rate per cent dividend?

26. State the three general problems illustrated in Exercises 23, 24, and 25.

27. What can I pay for 6% stock in order that I may make 4% upon my investment?

SOLUTION. 6% stock draws $6 per share of $100. Now $6 is 4% of $150. ∴ I can pay $150 per share, which is a quotation of 150.

28. If I pay $125 per share for 5% stock, what is the rate on my investment?

29. If I buy stock quoted at 80 and make 5% on my investment, what per cent stock did I buy?

30. State the three general problems illustrated in Exercises 27, 28, and 29.

31. What can I pay for 6% stock to make 5% upon my investment? To make 7%? To make $4\frac{1}{2}$%?

32. What can I pay for 15% stock to make 5%, $7\frac{1}{2}$%, 6%, or 8% on my investment?

33. The Chemical Bank Stock of New York has paid dividends of 90%. What could be paid for such stock to make 5% or 6% on the investment?

34. During the year 1900 the Standard Oil Co. declared dividends amounting to 48%. What could one pay for such stock to make 6% upon his investment?

35. If I pay $133\frac{1}{3}$ for 8% stock, what do I make on my investment?

36. If I own $4\frac{1}{2}\%$ United States bonds, par value \$8000, what is my annual income?

37. If I hold New York 6's, par value \$7500, what is my annual income?

38. If I pay \$7500 for Ohio 5's at 125, what is my annual income?

39. If a man exchanges 50 shares Pennsylvania 5's at 96 for United States $4\frac{1}{2}$'s at 120, how many United States $4\frac{1}{2}$'s did he receive? What *was* his annual income? What *is* it now? What are the inducements for such exchanges?

40. If I sell Alabama 5's, par value \$7500, at $96\frac{1}{8}$, and invest the proceeds in Mississippi $4\frac{1}{2}$'s at $79\frac{7}{8}$, brokerage each way $\frac{1}{8}$, do I change my annual income? How much?

Solution. $96\frac{1}{8} - \frac{1}{8} = 96$. $\therefore$ each share netted \$96. Now 75 shares at \$96 per share would net \$7200. $79\frac{7}{8} + \frac{1}{8} = 80$. $\therefore$ the total cost of 1 share of the Mississippi stock = \$80. \$7200 ÷ \$80 = 90. $\therefore$ there were 90 shares of Mississippi $4\frac{1}{2}$'s purchased.

The annual income on 75 Alabama 5's = \$375.

The annual income on 90 Mississippi $4\frac{1}{2}$'s = \$405.

The annual income is increased by \$30.

41. If I sell United States 4's, par value \$12,500, at $120\frac{5}{8}$ and invest sufficiently of the proceeds in Iowa 5's at 108 to insure an annual income of \$400 from them, and out of the remainder I purchase a house and lot for \$5000, how much money will I have left?

42. The net earnings of a company with a capital of \$480000 were \$33,000. Reserving \$4200 as a surplus, what per cent dividend can the company declare?

43. City bonds of 1900 bearing 5% were sold at $96\frac{1}{2}$. What per cent did the city pay on the money received?

44. What is the market value of 5% stock quoted at $85\frac{1}{2}$, brokerage $\frac{1}{8}$, whereby an annual income of \$2500 is realized?

45. How many shares of B. & O. stock must be bought at $102\frac{3}{4}$ and sold at $109\frac{1}{2}$, brokerage $\frac{1}{8}$ each way, that \$575 may be gained in the transaction?

46. How much money must a man in St. Joseph, Mo., invest in order to buy in New York 120 shares D., L. & W. stock at $156\frac{3}{4}$, brokerage $\frac{1}{8}$, New York Exchange $\frac{1}{5}\%$ premium?

47. Bought 600 shares Atchison for $ 59,100. What was the quotation price, brokerage $\frac{1}{8}$?

48. Sold 240 shares M., K. & T. for $ 19,200, brokerage $\frac{1}{8}$. What was the quoted price?

49. Sold 37 shares ($ 50) gas stock for $ 2039.62. Allowing no brokerage, what was the quoted price?

50. Bought stock at $197\frac{5}{8}$, brokerage $\frac{1}{8}$, and sold it at $197\frac{7}{8}$, brokerage $\frac{1}{8}$; in the meantime receiving a dividend of 6% on it, the net gain was $ 336. How many shares did I buy?

51. If an investor owns 4% stock purchased at 96, and 3% stock purchased at 80, in which stock has he the larger investment, if his income from each is $ 720 per year?

52. If a man receives an annual income of $ 225 from $4\frac{1}{2}\%$ stock and $ 240 from 2% stock, and he sells the former at 90 and the latter at 65, how much does he receive?

53. A person invested a certain sum of money in 6% stock at 90, and 3 times as much in 5% stock. If his income from the former was $ 240 and from the latter $2\frac{1}{2}$ times as much, what was the price of a share of the latter?

54. An investor drew quarterly dividends on his Michigan $4\frac{1}{2}$'s, amounting to $ 720. He afterward sold the bonds at $107\frac{1}{2}$, brokerage $\frac{1}{8}$. What were the proceeds of the sale?

55. A capitalist invested $ 18,000 as follows: $ 9500 in 5% high school bonds, at $94\frac{7}{8}$; $ 6500 in 7% gas stock, at $129\frac{7}{8}$; and the remainder in $5\frac{1}{2}\%$ court-house bonds at $99\frac{7}{8}$. Allowing brokerage $\frac{1}{8}$ in each instance, what is his total annual income?

56. Eighty shares of bank stock were purchased at 250. If this stock draws 16% annually, what is the income from them? What per cent is received on this investment?

57. If the stock of a street railway company consists of 10,000 shares, and the company issues $ 250,000 worth of 5% bonds at par, what per cent dividend can be declared on the stock of the company after paying the interest on the bonds, if the profits for the year were $ 42,500?

58. A has a farm which he rents at $ 411.45 per year. If he sells it for $ 8229 and invests the proceeds in 6% stock at 105, brokerage $\frac{1}{8}$, what is the change in his income per year?

59. If a lady wishes to secure an annual income of $ 1200, and finds 5% stock to suit her at $110\frac{1}{2}$, brokerage $\frac{1}{8}$, how much must she invest?

60. By selling 120 shares of 4% stock at $90\frac{1}{8}$ and investing the proceeds in other stock at $74\frac{1}{8}$, my income is changed $ 168, brokerage each way $\frac{1}{8}$. What is the entire income from my second investment? What was the rate per cent dividend on the second stock?

SERIES

A *series* is a succession of numbers formed according to some law. Thus 1, 2, 3, 4, 5, ⋯ ; 3, 7, 11, 15, ⋯ ; 86, 81, 76, 71, 66, 61, ⋯ ; 9, 5, 1, –3, –7, ⋯ ; 3, 6, 12, 24, ⋯ ; 192, 48, 12, 3, $\frac{3}{4}$, $\frac{3}{16}$, ⋯ ; and .272727 ⋯, are illustrations of series.

ARITHMETIC SERIES (Progression)

An arithmetic series is one in which any "term" is formed by adding a constant to, or subtracting it from, the preceding term.

The series 1, 5, 9, 13, ⋯, is formed by adding 4.

The series 3, $5\frac{1}{2}$, 8, $10\frac{1}{2}$, ⋯, is formed by adding $2\frac{1}{2}$.

The series 17, 14, 11, 8, 5, ⋯, is formed by subtracting 3.

The series 18, $15\frac{2}{3}$, $13\frac{1}{3}$, 11, ⋯, is formed by subtracting $2\frac{1}{3}$.

These are *arithmetic* series by definition.

The sum of the series 1, 5, 9, 13, 17, 21, 25, 30 = 121. It is easy to sum such a series, and it is easy to find the constant which is added to form it, as well as the 9th or 10th term say. But to sum the terms of the series 2, $3\frac{1}{2}$, 6, $8\frac{1}{2}$ to 1000 terms would be a laborious task without generalizing the subject.

We are concerned with the *first term*, the *last term*, the *number* of *terms*, the *constant*, and the *sum* of the series.

Let a = the first term,

l = the last term,

n = the number of terms,

d = the constant, called the *common difference*,

and s = the sum of the terms.

I. The nature of an arithmetic series is such that the second term exceeds the first by the common difference, the third exceeds the second by this difference, and the last term exceeds the last but one by this difference.

Hence, the second term $= a + d$, by definition,

the third term $= a + 2\,d$, by definition,

the fourth term $= a + 3\,d$, by definition,

$\vdots \qquad \vdots$

the eighth term $= a + 7\,d$, by definition,

$\vdots \qquad \vdots$

and the nth term $= a + (n-1)d$, by definition.

The nth term is the last term.

$$\therefore l = a + (n-1)d \tag{1}$$

Solve this equation for a,

$$\therefore a = l - (n-1)d \tag{2}$$

Solve for d, $$\therefore d = \frac{l-a}{n-1} \tag{3}$$

Solve for n, $$\therefore n = \frac{l-a}{d} - 1 \tag{4}$$

II. In the series 1, 4, 7, 10, 13, 16, the sum of the first and last terms equals the sum of the second and last but one, and equals the sum of the third and last but two. Write this series, and under it write the same series in the reverse order; thus,

$$\begin{array}{l} 1,\ 4,\ 7,\ 10,\ 13,\ 16 \\ \underline{16,\ 13,\ 10,\ 7,\ 4,\ 1} \\ 17,\ 17,\ 17,\ 17,\ 17,\ 17 = 6 \times 17 = 102. \end{array}$$

$\therefore$ the sum of the *two* series $= 102$; and the sum of the *one* series $= 51$.

$\therefore$ the sum of the double series is the sum of the first and last terms multiplied by the number of terms, and the sum of the single series is equal to the sum of the first and last terms multiplied by *half* the number of terms.

When generalized the sum of a series is:

$$s = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l$$

and

$$s = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a$$

$$\therefore 2s = (a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)+(a+l).$$

But there are n terms, each $=(a+l)$.

$$\therefore 2s = n(a+l),$$

$$\text{and } s = \frac{n}{2}(a+l) \qquad (5)$$

Solve for n, $\qquad \therefore n = \dfrac{2s}{a+l} \qquad (6)$

Solve for a, $\qquad \therefore a = \dfrac{2s}{n} - l \qquad (7)$

Solve for l, $\qquad \therefore l = \dfrac{2s}{n} - a \qquad (8)$

If l is not known, remember that $l = a + (n-1)d$.

$$\therefore s = \frac{n}{2}(a+l) = \frac{n}{2}(2a + (n-1)d) \qquad (9)$$

The first and fifth formulas are regarded as fundamental. From these others may be formed.

EXERCISES

1. Find the 12th term of the series 2, 6, 10, 14, ⋯.

2. Find the 15th term of the series 3, 24, 45, 66, ⋯.

3. Find the 40th term of the series 4, $7\frac{1}{2}$, 11, $14\frac{1}{2}$, ⋯.

4. Find the 18th term of the series 24, 21, 18, 15, ⋯.

5. Find the sum of 2, 9, 16, 23, ⋯ to 400 terms.

6. Find the sum of $1\frac{7}{8}$, 4, $6\frac{1}{8}$, $8\frac{1}{4}$, ⋯ to 68 terms.

7. Find s when $a = 24$, $d = 6$, and $n = 50$.

8. Find s when $a = 28$, $d = 2\frac{1}{4}$, and $n = 25$.

9. Find s when $a = \frac{1}{10}$, $d = \frac{1}{10}$, and $n = 800$.

10. A man walked 12 mi. the first day and increased this rate 2 mi. per day for 15 da. How far did he walk the last day? How far did he walk in the 16 da.?

11. A person had a gift of $75 the year of his birth, and each year thereafter he was to receive $5 more than the preceding year until he was 21 yr. of age. What did the whole gift amount to?

12. If a depositor should place $100 in a bank on Jan. 1, and the first of each succeeding month he should place $5 more than on the preceding month, what would the deposit amount to Dec. 1 of that year?

13. If a young man should enter an office at the age of 15 yr. at a salary of $300 a year, with an annual increase of $60, what would his salary be at the age of 25? of 40?

14. What total sum would the young man mentioned in Ex. 13 receive at the age of 50?

15. If a stone fall through 16.1 ft. the first second of time, 48.3 the next second, and 80.5 ft. the third, how deep is the shaft of a well if it takes a stone 6 sec. to reach the bottom?

16. Find the sum of the first 80 numbers which are divisible by 4.

GEOMETRIC SERIES (Progression)

A *geometric series* is one in which each term is formed by multiplying or dividing the preceding term by some constant.

The series 2, 8, 32, 128, $\cdots$ is formed by multiplying by 4.

The series 324, 108, 36, 12, $\cdots$ is formed by dividing by 3 or by multiplying by $\frac{1}{3}$.

These are geometric series, by definition.

In this series we are concerned with the *first term*, the *last term*, the *number* of terms, the *sum* of the terms, and the *constant*. It is customary to represent these by:

a, the first term,

l, the last term,

n, the number of terms,

s, the sum of the terms, and

r, the constant, which is called the *ratio*.

I. As in arithmetic series, it is necessary to generalize this subject. By the nature of a geometric series:

$$
\begin{array}{lll}
\text{the second term} & = ar, & \text{by definition,} \\
\text{the third term} & = ar^2, & \text{by definition,} \\
\text{the fourth term} & = ar^3, & \text{by definition,} \\
\vdots & \vdots & \\
\text{the tenth term} & = ar^9, & \text{by definition,} \\
\vdots & \vdots & \\
\text{the } n\text{th term} & = ar^{n-1}, & \text{by definition.}
\end{array}
$$

But the n*th* term is the last term,

$$\therefore l = ar^{n-1} \qquad (1)$$

Solve for a, $$\therefore a = \frac{l}{r^{n-1}} \qquad (2)$$

Solve for r, $$\therefore r = \sqrt[n-1]{\frac{l}{a}} \qquad (3)$$

II. The sum of the series 2, 6, 18, 54, 162, 486, is 728, where the ratio is 3. Three times this series is 6, 18, 54, 162, 486, 1458. The sum

of this series equals	$6+18+54+162+486+1458$
The sum of the given series equals	$2+6+18+54+162+486$
The difference equals	$-2 \qquad\qquad\qquad\qquad +1458$

$=1458-2$. $1458=3^6 \times 2$. $\therefore$ the difference is $2 \times 3^6 - 2$. Three times the given series less once the series equals two times the series.

$\therefore (3-1)$ sum $= 2 \times 3^6 - 2$. Hence, the sum of the given series $= \frac{2 \times 3^6 - 2}{3-1} = \frac{1456}{2} = 728.$

Generalizing by the use of the symbols:

$$s = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}.$$
$$rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$$
$$rs - s = ar^n - a;\ \text{or, } (r-l)s = ar^n - a.$$

$$\therefore s = \frac{ar^n - a}{r-l} \qquad (4)$$

Solve for r, $$\therefore r = \frac{s-a}{s-l} \qquad (5)$$

Solve for l, $$\therefore l = \frac{s(r-1)+a}{r} \qquad (6)$$

Solve for a, $$\therefore a = lr - s(r-1) \qquad (7)$$

EXERCISES

1. Find the 12th term of the series 2, 6, 18, 54, $\cdots$.

2. Find the 10th term of the series 3, 3^2, 3^3, 3^4, $\cdots$.

3. Find the 9th term of the series 1, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{64}$, $\cdots$.

4. Find the 11th term of the series 2, .2, .02, .002, $\cdots$.

5. Find the sum of the first 12 terms of the series in Ex. 1, and of the first 9 terms in Ex. 2.

6. Do the same for the first 14 terms of the series in Ex. 3, and for the first 20 terms in Ex. 4.

7. If a man were to buy 15 horses, paying 3¢ for the first horse, 9¢ for the second, and 27¢ for the third, what would they all cost him?

8. If the first term is 200, the ratio 1.36, and the number of terms 6, find the last term. The sum.

EXERCISES

1. What is the sum of the first 80 odd numbers?

2. What is the sum of the first 8 terms of the series 1, $\frac{1}{5}$, $\frac{1}{5}^2$, $\cdots$?

3. Insert 14 arithmetic terms between -15 and 30.

4. Find the nth term and the sum of the first n terms of the series 1, $\frac{3}{4}$, $\frac{1}{2}$, $\cdots$.

5. If an ivory ball is let fall a distance of 12 ft. and it rebounds 9.6 ft., and falling again rebounds 7.28 ft., and so on, through what distance will it have passed when it comes to rest?

6. If a father agrees to give his daughter \$1 at her marriage on New Year's Day, \$6 on the first day of February, \$36 on March first, and so on for 12 months, what sum is promised to the daughter?

CONTINUED FRACTIONS

Simple common fractions are reduced to equivalent fractions in the usual decimal form by dividing both terms of the fraction by such a number as will render the *denominator* unity, or 1. Thus:

$$\frac{1}{2}=\frac{.5}{1}=.5\,; \qquad \frac{3}{4}=\frac{3\div 4}{4\div 4}=\frac{.75}{1}=.75\,;$$

$$\frac{5}{8}=\frac{5\div 8}{8\div 8}=\frac{.625}{1}=.625\,; \quad \frac{2}{3}=\frac{2\div 3}{3\div 3}=\frac{.\dot{6}}{1}=.\dot{6}\,;$$

$$\frac{9}{11}=\frac{9\div 11}{11\div 11}=\frac{.\dot{8}\dot{1}}{1}=.\dot{8}\dot{1}\,; \quad \frac{4}{3}=\frac{4\div 3}{3\div 3}=\frac{1.\dot{3}}{1}=1.\dot{3}.$$

All of these reductions are based upon the principle that dividing both terms of a fraction by the same number does not change the value of the fraction.

By the above principle all simple fractions may be reduced so as to have unity or 1 for their *numerators*, and yet not change the value of the fractions. Thus:

$$\frac{2}{4}=\frac{1}{2}; \quad \frac{19}{57}=\frac{19\div 19}{57\div 19}=\frac{1}{3}; \quad \frac{3}{57}=\frac{3\div 3}{57\div 3}=\frac{1}{19};$$

$$\frac{3}{4}=\frac{3\div 3}{4\div 3}=\frac{1}{1+\frac{1}{3}}; \quad \frac{5}{6}=\frac{5\div 5}{6\div 5}=\frac{1}{1+\frac{1}{5}};$$

$$\frac{3}{8}=\frac{3\div 3}{8\div 3}=\frac{1}{2+\frac{2}{3}}=\frac{1}{2+\cfrac{1}{1+\frac{1}{2}}};$$

$$\frac{5}{8}=\frac{5 \div 5}{8 \div 5}=\frac{1}{1+\frac{3}{5}}=\frac{1}{1+\cfrac{1}{1+\frac{2}{3}}}=\frac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}};$$

$$\frac{49}{155}=\frac{1}{3+\frac{8}{49}}=\frac{1}{3+\cfrac{1}{6+\frac{1}{8}}};$$

$$\frac{68}{157}=\frac{1}{2+\frac{21}{68}}=\frac{1}{2+\cfrac{1}{3+\frac{5}{21}}}=\frac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\frac{1}{5}}}}.$$

DEFINITION. A fraction whose numerator is 1 and whose denominator is some integer plus a fraction whose numerator is 1, and whose denominator is an integer plus a fraction, and so on, is called a *continued fraction.*

Continued fractions were suggested to the world in a work published at Bologna in the early part of the seventeenth century. The author of this work is Cataldi, who found the square roots of numbers by means of continued fractions, but the plan was of little practical value.

On page 272 it is shown how several simple fractions may be reduced to equivalent continued fractions. The reduction of one more is here given:

$$\frac{28}{95}=\frac{1}{3+\frac{11}{28}}=\frac{1}{3+\cfrac{1}{2+\frac{6}{11}}}=\frac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\frac{5}{6}}}}=\frac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{5}}}}}.$$

In each move divide both the numerator and the denominator of the fraction to be reduced by the numerator of that fraction, and continue until the numerator of the last fraction is 1, whence the last *expression* is a continued fraction by the definition.

A continued fraction may be reduced to a simple fraction. Thus:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{3}}}}=\cfrac{1}{1+\cfrac{1}{1+\frac{1}{\frac{4}{3}}}}=\cfrac{1}{1+\cfrac{1}{1+\frac{3}{4}}}=\cfrac{1}{1+\frac{1}{\frac{7}{4}}}=\cfrac{1}{1+\frac{4}{7}}=\frac{1}{\frac{11}{7}}=\frac{7}{11}.$$

It is not attempted here to give a complete treatment of continuous fractions. For such treatment consult any good work on Algebra. Besides, there is very little in this subject that is of any value in everyday life. Concerning this and other subjects of as little value, De Morgan was led to say that they are "very useful for laying up grains of corn on the squares of a chessboard, ruining people by horseshoe bargains, and other approved problems."